Path categories and algorithms

Rick Jardine

GETCO 2015

April 8, 2015

n-cells

The n-**cell** $\square^n$ is the poset

$$\square^n = \mathcal{P}(\underline{n}),$$

the set of subsets of the totally ordered set $\underline{n} = \{1, 2, , \dots, n\}$.

There is a unique poset isomorphism

$$\mathcal{P}(\underline{n}) \xrightarrow{\cong} \mathbf{1}^{\times n},$$

where $\mathbf{1}$ is the 2-element poset $0 \leq 1$. Here,

$$A \mapsto (\epsilon_1, \dots, \epsilon_n)$$

where $\epsilon_i = 1$ if and only if $i \in A$. We use the ordering of $\underline{n}$.

The box category

Suppose that $A \subset B \subset \underline{n}$. The interval $[A, B] \subset \mathcal{P}(\underline{n})$ is defined by

$$[A, B] = \{C \mid A \subset C \subset B\}.$$

There are canonical poset maps

$$\mathcal{P}(\underline{m}) \cong \mathcal{P}(B - A) \xrightarrow{\cong} [A, B] \subset \mathcal{P}(\underline{n}).$$

where $m = |B - A|$. These compositions are the coface maps $d : \square^m \subset \square^n$.

There are also co-degeneracy map $s : \square^n \to \square^r$, which are again determined by subsets $A \subset \underline{n}$, where $|A| = r$, and such that $s(B) = B \cap A$.

The cofaces and codegeneracies are the generators for the **box category** $\square$ consisting of the posets $\square^n$, $n \geq 0$, subject to the standard cosimplicial identities.

Cubical sets and complexes

A **cubical set** is a functor $X : \square^{op} \to Sets$.

Typically $\square^n \mapsto X_n$, and X_n is the set of n-**cells** of X.

The collection of all such functors and natural transformations between them is the category c**Set** of cubical sets.

1) The **standard** n**-cell** $\square^n$ is the functor $\hom(\ ,\square^n)$ represented by $\square^n = \mathcal{P}(\underline{n})$.

2) A **finite cubical complex** is a subcomplex $K \subset \square^n$. It is completely determined by cells

$$\square^r \subset K \subset \square^n$$

where the composites are cofaces. A cell is **maximal** if r is maximal wrt these constraints.

Finite cubical complexes are **higher dimensional automata**.

Triangulation

There is a **triangulation functor**

$$| \cdot | : c\mathbf{Set} \to s\mathbf{Set}$$

$|\square^n| := B(\mathbf{1}^{\times n}) \cong (\Delta^1)^{\times n}$.

$B(C)$ is the **nerve** of a category C: $B(C)_n$ is the set

$$a_0 \to a_1 \to \cdots \to a_n$$

of strings of arrows of length n in C.

Example: $|\square^2|$:

$$\begin{array}{ccc} (0,1) & \longrightarrow & (1,1) \\ \uparrow & \nearrow & \uparrow \\ (0,0) & \longrightarrow & (1,0) \end{array}$$

The triangulation functor has a right adjoint,

$$S : s\mathbf{Set} \to c\mathbf{Set}$$

called the **singular** functor.

The path category

The nerve functor $B : \mathbf{cat} \to s\mathbf{Set}$ has a left adjoint

$$P : s\mathbf{Set} \to \mathbf{cat},$$

called the **path category** functor.

The path category $P(X)$ for X is the category generated by the 1-skeleton $\mathrm{sk}_1(X)$ (a graph), subject to some relations:

1) $s_0(x)$ is the identity morphism for all vertices $x \in X_0$,

2) the triangle

$$\begin{array}{ccc} \sigma_0 & \xrightarrow{d_2(\sigma)} & \sigma_1 \\ & \searrow^{d_1(\sigma)} & \downarrow{\scriptstyle d_0(\sigma)} \\ & & \sigma_2 \end{array}$$

commutes for all 2-simplices $\sigma : \Delta^2 \to X$ of X.

Execution paths

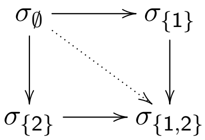
Suppose that $K \subset \square^n$ is an HDA, with states (vertices) x, y. Then

$$P(|K|)(x, y)$$

is the set of execution paths from x to y. We want to compute these.

$P(K) := P(|K|)$ is the path category of the complex K.

It can be defined directly for K: it is generated by the graph $\mathrm{sk}_1(K)$, subject to the relations given by $s_0(x) = 1_x$ for vertices x, and by forcing the commutativity of



for each 2-cell $\sigma : \square^2 \subset K$ of K.

Preliminary facts

Lemma 1.

1) $\mathrm{sk}_2(X) \subset X$ *induces* $P(\mathrm{sk}_2(X)) \cong P(X)$.
2) $\epsilon : P(BC) \to C$ *is an isomorphism for all small categories* C.

The path 2-category

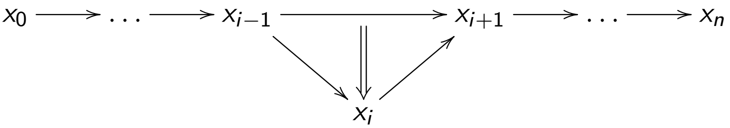
$L =$ finite simplicial complex. "$P(L)$ is the path component category of a 2-category $P_2(L)$."

$P_2(L)$ consists of categories $P_2(L)(x,y)$, one for each pair of vertices $x, y \in L$.

The objects (1-cells) are paths of non-deg. 1-simplices

$$x = x_0 \to x_1 \to \cdots \to x_n = y$$

of L. The morphisms of $P_2(L)(x,y)$ are composites of the pictures



where the displayed triangle bounds a non-deg. 2-simplex.

Compositions are functors

$$P_2(L)(x,y) \times P_2(L)(y,z) \to P_2(L)(x,z)$$

defined by concatenation of paths.

Theorem 2.

$P_2(L)$ is a "resolution" of the path category $P(L)$ in the sense that there is an isomorphism

$$\pi_0 P_2(L) \cong P(L).$$

$\pi_0 P_2(L)$ is the **path component category** of $P_2(L)$. Its objects are the vertices of L, and

$$\pi_0 P_2(L)(x,y) = \pi_0(BP_2(L)(x,y)).$$

The algorithm

Here's an algorithm for computing $P(L)$ for $L \subset \Delta^N$, in outline:

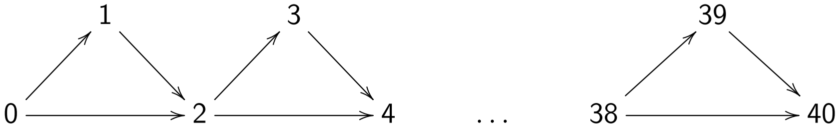
1) Find the 2-skeleton $\mathrm{sk}_2(L)$ of L (vertices, 1-simplices, 2-simplices).
2) Find all paths (strings of 1-simplices)

$$\omega : v_0 \xrightarrow{\sigma_1} v_1 \xrightarrow{\sigma_2} \dots \xrightarrow{\sigma_k} v_k$$

   in L.
3) Find all morphisms in the category $P_2(L)(v, w)$ for all vertices $v < w$ in L (ordering in Δ^N).
4) Find the path components of all $P_2(L)(v, w)$, by approximating path components by full connected subcategories, starting with a fixed path ω.

An example

Let $L \subset \Delta^{40}$ be the subcomplex



This is 20 copies of the complex $\partial\Delta^2$ glued together. There there are 2^{20} morphisms in $P(L)(0,40)$.

Moral: The size of the path category $P(L)$ can grow exponentially with L.

The code for this example runs on a desktop with at least 5 GB of memory. The listing of paths consumes 2 GB of disk.

Suppose that $L \subset K \subset \Delta^N$ defines L as a subcomplex of K.

L is a **full subcomplex** of K if the following hold:

1) L is path-closed in K, in the sense that, if there is a path

$$v = v_0 \to v_1 \to \cdots \to v_n = v'$$

in K between vertices v, v' of L, then all $v_i \in L$,

2) if all the vertices of a simplex $\sigma \in K$ are in L then the simplex σ is in L.

Lemma 3.

Suppose that L is a full subcomplex of K. Then the functor $P(L) \to P(K)$ is fully faithful.

Examples

- $\partial\Delta^2 \overset{d^0}{\subset} \Lambda^3_0$ and $\partial\Delta^2 \overset{d^3}{\subset} \Lambda^3_3$ are full subcomplexes.
- Suppose that $i \leq j$ in $\mathbf{N}$. $K[i,j]$ is the subcomplex of K such that $\sigma \in K[i,j]$ if and only if all vertices of σ are in the interval $[i,j]$ of vertices v such that $i \leq v \leq j$. $K[i,j]$ is a full subcomplex of K.
- Suppose that $v \leq w$ are vertices of K. Let $K(v,w)$ be the subcomplex of K consisting of simplices whose vertices appear on a path from v to w. $K(v,w)$ is a full subcomplex of K.

One can construct $K(v,w)$ from $K[v,w]$ by deleting sources and sinks.

Say that a vertex v is a **source** of K if there are no 1-simplices $u \to v$ in K. The vertex v is a **sink** if there are no 1-simplices $v \to w$ in K.

Suppose that $K \subset \square^n$ is a cubical complex. Say that a vertex x is a **corner** of K if it belongs to only one maximal cell.

Lemma 4 (Misamore).

Suppose that x is a corner of K, and let K_x be the subcomplex of cells which do not have x as a vertex. Then the induced functor

$$P(K_x) \to P(K)$$

is fully faithful.

There are two steps in the proof [3]:

- Suppose that x is a vertex of the cell $\square^r$ and let $\square^r_x \subset \square^r$ be the subcomplex of cells which do not have x as a vertex. Then $P(\square^r_x) \to P(\square^n)$ is fully faithful.

- Suppose that x is a corner of K, and that x is a vertex of a maximal cell $\square^r \subset K$. Let $K_x \subset K$ be the subcomplex whose cells do not have x as a vertex. Then the diagram

$$\begin{array}{ccc} P(\square^r_x) & \longrightarrow & P(K_x) \\ \downarrow & & \downarrow \\ P(\square^r) & \longrightarrow & P(K) \end{array}$$

  is a pushout, so that $P(K_x) \to P(K)$ is fully faithful.

This uses an assertion of Fritsch and Latch [1] that fully faithful functors are closed under pushout.

Examples

1) The cubical horn

$$\begin{array}{ccc} (0,1) & \longrightarrow & (1,1) \\ \uparrow & & \uparrow \\ (0,0) & & (1,0) \end{array}$$

has a sink but no corners.

2) The Swiss flag

$$\begin{array}{ccccccc} \bullet & \to & \bullet & \to & \bullet & \to & \bullet \\ \uparrow & \searrow & \uparrow & & \uparrow & \searrow & \uparrow \\ \bullet & \to & * & & * & \to & \bullet \\ \uparrow & & & & & & \uparrow \\ \bullet & \to & * & & * & \to & \bullet \\ \uparrow & \searrow & \uparrow & & \uparrow & \searrow & \uparrow \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet \end{array}$$

has 6 corners, 1 sink, 1 source.

Going beyond

The algorithms that we have depend on having an entire HDA in storage, in a computer system that is powerful enough to analyze it.

We want local to global methods to study large (aka. "infinite") models with patching techniques.

The time variable

Suppose that $K \subset \square^N$. There is a poset map

$$\mathcal{P}(\underline{N}) \xrightarrow{t} \mathbb{Z}_{\geq 0} \subset \mathbb{Z},$$

with $F \mapsto |F|$. There are induced simplicial set maps

$$|K| \subset |\square^N| = B\mathcal{P}(\underline{N}) \xrightarrow{t} B\mathbb{Z}_{\geq 0} \subset B\mathbb{Z}.$$

In a standard HDA, the state represented by F is reached only after $|F|$ clock ticks. We thus have a fibring of the triangulated HDA over a time poset.

The pre-images of the intervals $[i,j] \subset \mathbb{Z}_{\geq 0}$ give a coarse sense of locality for $|K|$.

More generally, one might ask for a lattice homomorphism

$$\phi : \mathcal{P}(\underline{N}) \to Q$$

with ϕ is determined by the maps $\phi(\emptyset) \to \phi(\{i\})$ for all $i \in \underline{N}$.

Smallest elements and intervals

Suppose that A, B are subsets of $\underline{n}$. Say that A consists of **smallest elements** outside B if

1) $A \cap B = \emptyset$, and
2) if $i \leq j$ for some $j \in A$ and $i \notin B$, then $i \in A$.

Example: $A =$ totally ordered finite set, and $[C, D] \subset \mathcal{P}(A)$ an interval, with $\psi : \mathcal{P}(D - C) \to \mathcal{P}(A)$ st $E \mapsto C \sqcup E$.
ψ is completely determined by a string of subsets

$$C = A_0 \subset A_1 \subset \cdots \subset A_{r-1} \subset A_r = D,$$

$$A_{i+1} = A_i \sqcup \{x_{i+1}\},$$

and x_{i+1} is the smallest element of D which is outside A_i. Then

$$D \cong C \sqcup \{x_1, \dots, x_r\}$$

via a bijection which is ordered on each summand (ie. a shuffle).

$B =$ totally ordered finite set. A **refinement** R in B is a string

$$B_0 \subset B_1 \subset \cdots \subset B_r$$

of subsets of B such that $B_{i+1} - B_i$ consists of smallest elements of B which are outside B_i for $0 \leq i \leq r-1$.

Every refinement determines a poset morphism

$$\phi_R : \mathcal{P}(\underline{r}) \to \mathcal{P}(B)$$

such that $\phi_R(\emptyset) = B_0$ and $\phi_R(\{i\}) = B_0 \sqcup (B_{i+1} - B_i)$, and more generally

$$\phi_R(F) = B_0 \sqcup (\sqcup_{j \in F}\ \phi(\{j\}))$$

for all subsets $F \subset \underline{r}$. In particular, $\phi(\underline{r}) = B_r$.

The map ϕ_R is a refinement of $\square^r = \mathcal{P}(\underline{r})$ in a bigger box $\mathcal{P}(B)$.

Properties

1) Refinements are closed under composition (successive cofaces in a nerve).

2) Every refinement $\mathcal{P}(\underline{r}) \to \mathcal{P}(B)$ is a refinement of a unique face (interval) of $\mathcal{P}(B)$.

A refinement is a generalized time variable.

3) Every refinement R in B and every cell $d : \mathcal{P}(\underline{k}) \to \mathcal{P}(\underline{r})$ together determine a unique commutative diagram

$$\begin{array}{ccc} \mathcal{P}(\underline{k}) & \xrightarrow{\phi_R} & \mathcal{P}(\underline{k'}) \\ d \downarrow & & \downarrow d' \\ \mathcal{P}(\underline{r}) & \xrightarrow[\phi_R]{} & \mathcal{P}(B) \end{array}$$

where d and d' are cells.

4) Every subcomplex $K \subset \square^r$ has a refinement $K_R \subset \square^{|B|}$.

There is a canonical diagram of simplicial set maps

$$\begin{array}{ccccc} |K| & \longrightarrow & |K_R| & & \\ \downarrow & & \downarrow & & \\ B\mathcal{P}(\underline{r}) & \rightarrow & B\mathcal{P}(B) & & \\ & & & \searrow & \\ & & & & B\mathbb{Z}_{\geq 0} \end{array}$$

Starting knowledge of a system could be an initial HDA $K_0 \subset \square^{n_0}$, but there could be successive refinements

$$\begin{array}{ccccc} |K_0| & \longrightarrow & |K_1| & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \\ B\mathcal{P}(\underline{n}_0) & \longrightarrow & B\mathcal{P}(\underline{n}_1) & \longrightarrow & \dots \end{array}$$

Examples

1) $\emptyset \subset \{1,2\}$ is a refinement of $\underline{1}$ in $\underline{2}$. The corr. poset map $\mathbf{1} \to \mathbf{1}^{\times 2}$ is the diagonal 1-simplex

$$(0,0) \to (1,1).$$

2) The string $\emptyset \subset \{1,2\} \subset \{1,2,3,4\}$ is a refinement of $\underline{2}$ in $\underline{4}$. The corresponding poset map $\mathbf{1}^{\times 2} \to \mathbf{1}^{\times 4}$ is defined by the picture

$$\begin{array}{ccc} (0,0,0,0) & \longrightarrow & (1,1,0,0) \\ \downarrow & & \downarrow \\ (0,0,1,1) & \longrightarrow & (1,1,1,1) \end{array}$$

This picture also defines the subdivision $\mathrm{sd}(\square^2)$ of $\square^2$ in $\square^4$.

References

Rudolf Fritsch and Dana May Latch.
Homotopy inverses for nerve.
Math. Z., 177(2):147–179, 1981.

J. F. Jardine.
Path categories and resolutions.
Homology Homotopy Appl., 12(2):231–244, 2010.

Michael D. Misamore.
Computing path categories of finite directed cubical complexes.
Applicable Algebra in Engineering, Communication and Computing, pages 1–14, 2014.