Applied Algebraic Topology
Concurrency and Direction

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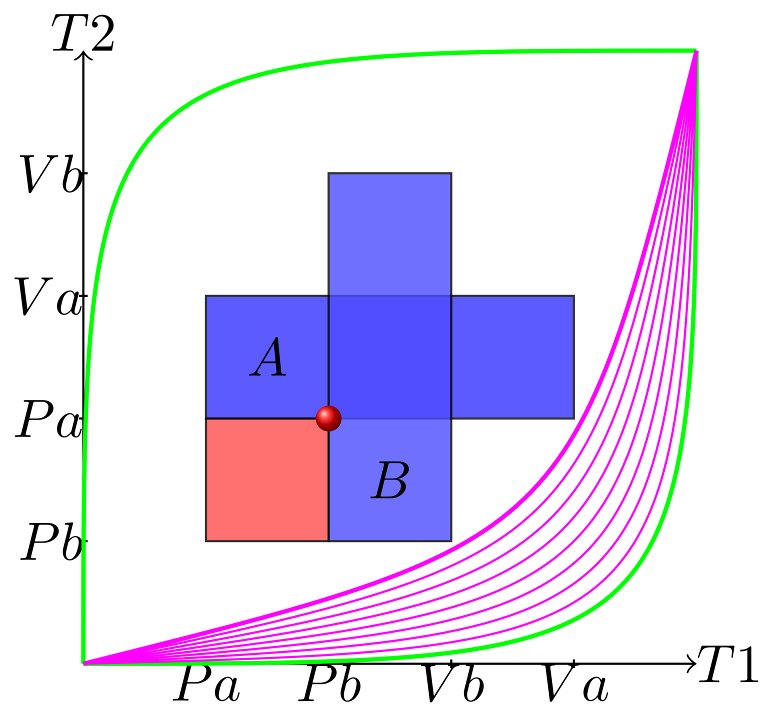
From concurrency to geometry

We attack problems in concurrent computing with geometry/topology. In order to do that, we develop new mathematics.
The geometric model of concurrency:
Interaction is through shared resources: Memory, printers,... with limited capacity.
Threads are given by a sequence of locks Pr and releases Vr of a resource r.
E.g.

$$PaPb(PcVc)^*VaVb$$

The geometric model of each thread is a graph.
The model of threads in parallel is a product of their graphs - with holes.

Shared resources - Conflict



When concurrent processes access shared resources, conflicts may arise.

The geometric model of a conflict is a hole.
An execution is a directed path.
Many executions are equivalent.
Some regions are unsafe - no directed path leaves.

Questions from concurrency

How many *different executions* are there?
Aim: Verification only needed for one in each equivalence class.
Locate *deadlocks* and avoid them.
Find efficient algorithms to do so.

Executions are equivalent (give rise to the same output) if the corresponding paths can be continuously deformed into each other.
Geometry modulo deformation = Algebraic topology .
Geometry modulo deformation *respecting direction* $\rightsquigarrow$ Directed topology, a new mathematical area.

Who are we ?


Lisbeth Fajstrup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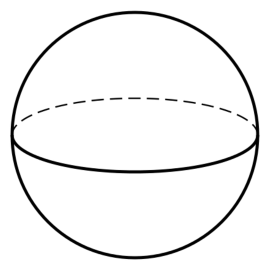
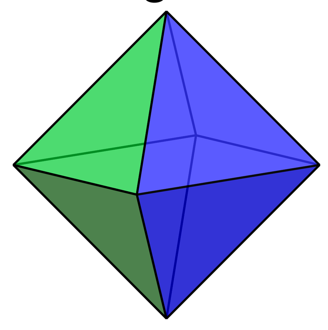
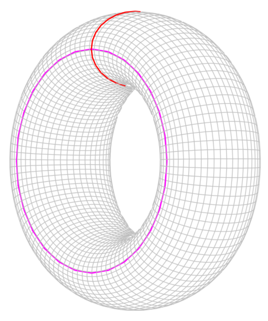
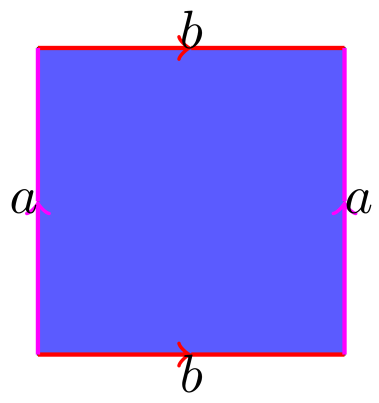
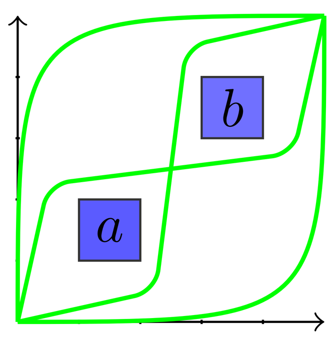
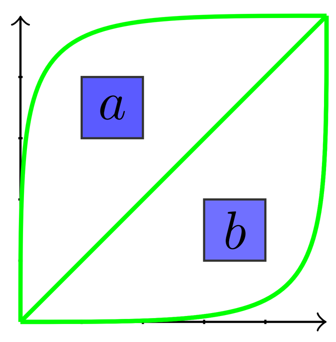
Algebraic Topology

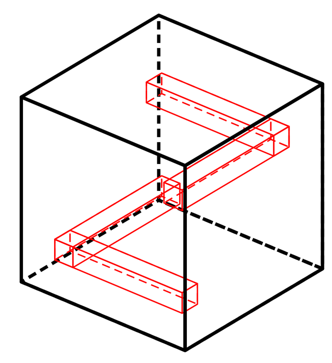
Geometry - Shape	Combinatorial models	Algebra - Invariants	
	Simplices or cubes Gluing rules	Loops and holes	Euler Characteristic, χ $\Sigma_{i=0}^{n}(-1)^n\#\{n\text{-simplices}\}$
		One 2-dim'l hole All loops contractible	$\chi = 6 - 12 + 8 = 2$
		Two 1 dim'l holes. Loops determined by # turns around each hole Loops $\sim \mathbb{Z} \oplus \mathbb{Z}$	$\chi = 0$

Directed Algebraic Topology - a new mathematical area

There may be no directed loops. Instead we look at (directed) paths from beginning to end. As a consequence, the algebraic structure is not as nice. No inverses!







Directed paths:	4 classes	3 classes	2 classes
Undirected paths:	$a^{k_1}b^{m_1}a^{k_2}b^{m_2}\dots a^{k_l}b^{m_l},\ k_i, m_i \in \mathbb{Z}$ k_1 turns around hole a then m_1 turns around hole b etc. The free group on two generators.		One class

Work in Dir.Alg.Top

Directed algebraic topology is *studied as a mathematical field.*
There are connections to and tools from other areas - Category Theory, Configurations Spaces, Lattice Theory, Combinatorics.
Some properties and subjects investigated are *dicoverings, combinatorial directed structures, spaces of directed paths, directed homology, directed components, and other invariants.*

Other applications

Topological Data Analysis, the analysis of the shape of data, is related and connected to our work. The upshot is to look not only for clusters of data, but also data which is "looped" - has periodicity. Or has other underlying geometric features. The toolbox contains a new development in algebraic topology, Persistent Homology.
Distributed Computing: Fault tolerant protocols. Consensus problems. Impossibility results. Studied and proven using invariants from algebraic topology.

Collaborators

We collaborate with researchers from France, Italy, Poland, Israel, Slovenia, USA,... .
We are invited to give talks and tutorials at conferences both in mathematics and computer science.
We are part of the research network ACAT under the ESF.

Algorithms

We have developed algorithms to analyse PV-programs to find:
Deadlocks and unsafe areas
Equivalence classes of executions.
Implemented by our French and Polish collaborators in Alcool, a static analyzer used for verification. Using CrHom, a homology software.

Publications (excerpt)

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Raussen, M., (2012). Simplicial models for Trace Spaces II. General Higher Dimensional Automata. *Algebr.Geom. Topol. 12*(3), 1745–1765.

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Fajstrup, L.,**Goubault, E.**, **Raussen, M.** (2006). Algebraic Topology and Concurrency. *Theor. Comput. Sci. 357*, 241–278.