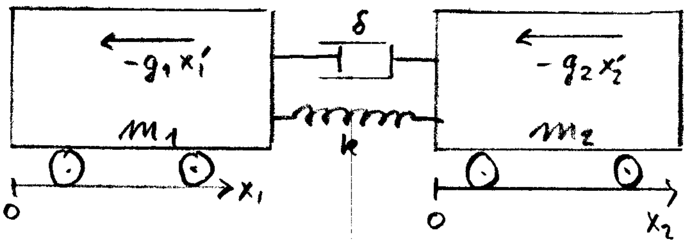
# Opgave 1, eksamen juni 2006

(1) Med vognene i position $x_1$ hhv. $x_2$ er kompressionen af fjederen lig $x_1 - x_2$, så




$$m_1 x_1'' = -k(x_1 - x_2) - \delta(x_1' - x_2') - g_1 x_1'$$
$$m_2 x_2'' = -k(x_2 - x_1) - \delta(x_2' - x_1') - g_2 x_2'$$

Indføres hjælpefunktionerne $x_3(t) = x_1'(t)$ og $x_4(t) = x_2'(t)$ fås

$$\begin{aligned}
x_1'(t) &= x_3(t)\\
x_2'(t) &= x_4(t)\\
x_3'(t) &= -\frac{k}{m_1}x_1(t) + \frac{k}{m_1}x_2(t) - \frac{\delta+g_1}{m_1}x_3(t) + \frac{\delta}{m_1}x_4(t)\\
x_4'(t) &= \frac{k}{m_2}x_1(t) - \frac{k}{m_2}x_2(t) + \frac{\delta}{m_2}x_3(t) - \frac{\delta+g_2}{m_2}x_4(t)
\end{aligned}$$

For vektorfunktionen $\bar{x}(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ fås heraf

$$\frac{d\bar{x}}{dt} = \begin{bmatrix} x_1'(t)\\ x_2'(t)\\ x_3'(t)\\ x_4'(t)\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -k/m_1 & k/m_1 & -\frac{\delta+g_1}{m_1} & \frac{\delta}{m_1}\\ k/m_2 & -k/m_2 & \frac{\delta}{m_2} & -\frac{\delta+g_2}{m_2}\end{bmatrix}\cdot\begin{bmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix} = A\cdot\bar{x}(t).$$

(2) For $m_1 = m_2 = 1$, $k = 1$, $\delta = 1$, $g_2 = 1$ og $g_2 = 0$ fås

$$A = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 1 & -1 & 1\\ 1 & -1 & 1 & -2\end{bmatrix}$$

Denne matrix har det karakteristiske polynomium

$$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & 1 & 0\\ 0 & -\lambda & 0 & 1\\ -1 & 1 & -1-\lambda & 1\\ 1 & -1 & 1 & -2-\lambda\end{vmatrix} = \begin{vmatrix} -\lambda & 0 & 1 & 0\\ 0 & -\lambda & 0 & 1\\ 0 & 0 & -\lambda & -1-\lambda\\ 1 & -1 & 1 & -2-\lambda\end{vmatrix}$$

$$= \begin{vmatrix} 0 & -\lambda & 1+\lambda & -2\lambda-\lambda^2\\ 0 & +\lambda & 0 & -1\\ 0 & 0 & +\lambda & +1+\lambda\\ 1 & -1 & 1 & -2-\lambda\end{vmatrix} = (-1)^5\cdot 1\cdot\begin{vmatrix} -\lambda & 1+\lambda & -2\lambda-\lambda^2\\ \lambda & 0 & -1\\ 0 & \lambda & 1+\lambda\end{vmatrix} = -\begin{vmatrix} 0 & 1+\lambda & -1-2\lambda-\lambda^2\\ \lambda & 0 & -1\\ 0 & \lambda & 1+\lambda\end{vmatrix}$$

$$= -(-1)^3\cdot\lambda\cdot\begin{vmatrix} 1+\lambda & -1-2\lambda-\lambda^2\\ \lambda & 1+\lambda\end{vmatrix} = \lambda((1+\lambda)^2 + \lambda + 2\lambda^2 + \lambda^3)$$

$$= \lambda((1+\lambda)^2 + \lambda(1+2\lambda+\lambda^2)) = \lambda((1+\lambda)^2 + \lambda(1+\lambda)^2) = \lambda(1+\lambda)^3.$$

Rødderne heri er åbenbart

$\lambda = 0$, multiplicitet 1
$\lambda = -1$, multiplicitet 3.

For $v_0 = (1,1,0,0)$ ses direkte at

$$(A - \lambda_0 I)v_0 = (A - 0 \cdot I)v_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Derfor er $v_0$ <u>egenvektor</u> hørende til $\lambda_0 = 0$.

For $\lambda_1 = -1$ løses ligningen $(A - \lambda_1 I)v = 0$. Dvs.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Disse ligninger er ækvivalente med

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Derfor er $v_1 = 0$, så $v_3 = -v_1 = 0$.
Og $v_2 + v_4 = 0$.

Løsningsmængden er således

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = t \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

$\vec{v}_1 = (0,1,0,-1)$ er derfor en basis for egenrummet hørende til $\lambda_1 = -1$ (dvs. alle andre egenvektorer hørende til $\lambda_1$ er lineært afhængige af $\vec{v}_1$).

Således er egenværdien $\lambda_1 = -1$ defekt, med

$$\underline{d = 3 - 1 = 2.}$$

(3) Det ses direkte at

$$(A - \lambda_1 I)^{d+1} = (A + I)^3 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Vektoren $\vec{v}_3 = (1,0,0,0)$ opfylder klart $(A+I)^3 \vec{v}_3 = \vec{0}$.
Så indføres

$$\vec{v}_2 = (A + I)\vec{v}_3 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{v}_1 = (A + I)\vec{v}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

Dermed er $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ en kæde af generaliserede egenvektorer hørende til $\lambda_1 = -1$.

(Opgave 1, fortsat) 

Den generelle løsning til $\vec{x}' = A \cdot x$ har nu formen, med konstanter $c_0, c_1, c_2, c_3$,

$$\vec{X}(t) = c_0 e^{0 \cdot t} \vec{v}_0 + c_1 e^{(-1)t} \vec{v}_1 + c_2 e^{-t}(t \cdot \vec{v}_1 + \vec{v}_2) + c_3 e^{-t}(\tfrac{1}{2}t^2 \vec{v}_1 + t\vec{v}_2 + \vec{v}_3)$$

$$= c_0 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_1 e^{-t} \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} +1 \\ t \\ -1 \\ -t+1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} t+1 \\ \frac{1}{2}t^2 - t \\ 0 \\ -\frac{1}{2}t^2 + t \end{bmatrix}$$

(4) Da begyndelsesbetingelsen er $\vec{X}(0) = (0, 0, V_0, 0)$ løses

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ V_0 \\ 0 \end{bmatrix}$$

Successivt ses at
$c_2 = -V_0$, $c_1 = -V_0$
$c_0 = V_0$, $c_3 = 0$.

Formlen for $\vec{x}(t)$ giver så

$$x_1(t) = V_0(1 - e^{-t})$$

$$x_2(t) = V_0(1 - e^{-t} - te^{-t})$$

Vognenes afstand er, for $t > 0$,

$$x_2(t) - x_1(t) = -V_0 \cdot t \cdot e^{-t} < 0$$

så deres indbyrdes afstand er mindre end for $t = 0$.
[NB! I hvilestillingen er $x_2(0) - x_1(0) = 0$, som opnås ved at måle positionerne langs forskudte akser; jvf. figuren.]

Opgave 2 (a) Varmeledningsligningen: $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$.

Med stationær temperaturfordeling $w(x)$ er det klart at

$$k \frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial t} = 0$$

$w(x)$ er derfor et polynomium af grad $\leq 1$ som opfylder

$$w(0) = A \quad ; \quad w(L) = B.$$

Dette polynomium er oplagt givet ved

$$w(x) = A + \frac{B-A}{L} \cdot x$$

(opgave 2, fortsat) 

(2) Den tidsafhængige del af temperaturfordelingen

$$v(x,t) = u(x,t) - w(x)$$

opfylder, da w's anden afledte er nul,

$$\underline{\underline{\frac{\partial v}{\partial t}}} = \frac{\partial}{\partial t}(u-w) = \frac{\partial u}{\partial t} - 0$$

$$= k\frac{\partial^2 u}{\partial x^2} \qquad = k\frac{\partial^2}{\partial x^2}(v+w) = k\frac{\partial^2 v}{\partial x^2} + k\frac{\partial^2 w}{\partial x^2} = \underline{\underline{k\frac{\partial^2 v}{\partial x^2}}}$$

$$v(0,t) = u(0,t) - w(0) = A - A = 0$$

$$v(L,t) = u(L,t) - w(L) = B - B = 0$$

$$v(x,0) = u(x,0) - w(x) = f(x) - w(x)$$

Som ønsket.

(3) Idet $v(x,t)$ løser varmeledningsligningen med **homogen** Dirichletrandbetingelse, er

$$v(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-n^2 \cdot \frac{\pi^2}{L^2} \cdot k \cdot t\right) \sin\left(n \cdot \frac{\pi}{L} \cdot x\right)$$

$$b_n = \frac{2}{L}\int_0^L (f(x) - w(x)) \sin\left(n \cdot \frac{\pi}{L} \cdot x\right) dx, \quad n=1, \ldots$$

Da $u(x,t) = v(x,t) + w(x)$ fås, for de samme $b_n$-er,

$$\underline{\underline{u(x,t) = A + \frac{B-A}{L} \cdot x + \sum_{n=1}^{\infty} b_n \exp\left(-n^2 \cdot \frac{\pi^2}{L^2} \cdot k \cdot t\right) \sin\left(n \cdot \frac{\pi}{L} \cdot x\right)}}$$

(4) Her er $w(x) = \frac{100}{50} \cdot x = 2x$ mens $f(x) \equiv 0$, så

$$b_n = \frac{2}{50}\int_0^{50} (-2x) \sin\left(n \cdot \frac{\pi}{50} \cdot x\right) dx$$

$$= \frac{4}{50}\int_0^{50} x \cdot \frac{d}{dx}\cos\left(n \cdot \frac{\pi}{50} \cdot x\right) \cdot \frac{50}{n \cdot \pi}\, dx$$

$$= \frac{4}{n \cdot \pi}\left[x \cdot \cos\left(n\frac{\pi}{50}x\right)\right]_{x=0}^{x=50} - \frac{4}{n \cdot \pi}\int_0^{50} 1 \cdot \cos\left(n \cdot \frac{\pi}{50} \cdot x\right) dx$$

$$= \frac{200}{n \cdot \pi} \cdot (-1)^n - 0.$$

$$u(25,60) = 2 \cdot 25 + \sum_{n=1}^{\infty} (-1)^n \frac{200}{n \cdot \pi} \exp\left(-n^2 \cdot \frac{\pi^2}{2500} \cdot 60\right) \sin\left(n \cdot \frac{\pi}{50} \cdot 25\right)$$

$$= 50 - 47{,}75\ldots + \ldots$$

$$\simeq 2{,}2478\ldots$$

## Opgave 3

(1)
$$\frac{\partial u}{\partial t} = -\frac{3}{2}\left(1+(\bar{a}\cdot\bar{x}-t)^2\right)^{-5/2}\cdot 2(\bar{a}\cdot\bar{x}-t)(-1)$$
$$\frac{\partial u}{\partial x_j} = -\frac{3}{2}\left(1+(\bar{a}\cdot\bar{x}-t)^2\right)^{-5/2}\cdot 2(\bar{a}\cdot\bar{x}-t)\cdot a_j$$

(2)
$$\frac{\partial^2 u}{\partial t^2} = \frac{3}{2}\cdot\frac{5}{2}\cdot\left(1+(\bar{a}\cdot\bar{x}-t)^2\right)^{-7/2}\cdot\left(2(\bar{a}\cdot\bar{x}-t)(-1)\right)^2 - \frac{3}{2}\cdot\left(1+(\bar{a}\cdot\bar{x}-t)^2\right)^{-5/2}\cdot 2$$

$$\frac{\partial^2 u}{\partial x_j^2} = \frac{3}{2}\cdot\frac{5}{2}\left(1+(\bar{a}\cdot\bar{x}-t)^2\right)^{-7/2}\cdot\left(2(\bar{a}\cdot\bar{x}-t)\cdot a_j\right)^2 - \frac{3}{2}\cdot\left(1+(\bar{a}\cdot\bar{x}-t)^2\right)^{-5/2}\cdot 2a_j^2$$
$$= a_j^2\cdot\frac{\partial^2 u}{\partial t^2}$$

Man har derfor at
$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2} = a_1^2\frac{\partial^2 u}{\partial t^2}+a_2^2\frac{\partial^2 u}{\partial t^2} = (a_1^2+a_2^2)\frac{\partial^2 u}{\partial t^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

(3). Ved kædereglen fås om $u(x,t) = F(\bar{a}\cdot\bar{x}\pm t)$ at
$$\frac{\partial u}{\partial t} = F'(\bar{a}\cdot\bar{x}\pm t)\cdot(\pm 1) \qquad \frac{\partial^2 u}{\partial t^2} = F''(\bar{a}\cdot\bar{x}\pm t)(\pm 1)^2$$
$$\frac{\partial u}{\partial x_1} = F'(\bar{a}\cdot\bar{x}\pm t)\cdot a_1 \qquad \frac{\partial^2 u}{\partial x_1^2} = F''(\bar{a}\cdot\bar{x}\pm t)\,a_1^2$$

og tilsvarende for $\frac{\partial^2 u}{\partial x_2^2}$ så

$$c^2\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}\right) = c^2F''(\bar{a}\cdot\bar{x}\pm t)a_1^2 + c^2F''(\bar{a}\cdot\bar{x}\pm t)a_2^2$$
$$= c^2(a_1^2+a_2^2)F''(\bar{a}\cdot\bar{x}\pm t)$$
$$= F''(\bar{a}\cdot\bar{x}\pm t)$$
$$= \frac{\partial^2 u}{\partial t^2}$$

Dermed er $u(x,t) = F(\bar{a}\cdot\bar{x}\pm t)$ en løsning til bølgeligningen
(endda en <u>plan bølge</u>, der udbredes i $\bar{a}$'s retning)