ET, MP <u>Matematik 3</u> 25/1/2013

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<u>OPG 1:</u> $y''(t)+4y'(t)-5y(t)=e^{-t}$, $y(0)=-1$, $y'(0)=4$

Den <u>Laplacetransformerede</u> $Y(s)=\mathcal{L}(y)$ opfylder

$$\mathcal{L}(y'')=s^2Y(s)-sy(0)-y'(0)$$
$$\mathcal{L}(y')=sY(s)-y(0)$$

så $\mathcal{L}$ fører ligningen over i

$$s^2Y(s)-sy(0)-y'(0)+4(sY(s)-y(0))-5Y(s)=\mathcal{L}(e^{-t}),$$

og da $\mathcal{L}(e^{-t})=\frac{1}{s-(-1)}=\frac{1}{s+1}$,

$$s^2Y(s)+4sY(s)-5Y(s)=\frac{1}{s+1}+sy(0)+y'(0)+4y(0)$$

dvs.

$$(s^2+4s-5)\cdot Y(s)=\frac{1}{s+1}-s$$

Da s^2+4s-5 har rødderne $s=1$ og $s=-5$, så fås

$$Y(s)=\frac{1}{s+1}\cdot\frac{1}{s-1}\cdot\frac{1}{s+5}-\frac{s}{(s-1)(s+5)}$$

Iflg. formel (+) i opgaven og Kreyzig [Tabel 6.9.12]

$$\mathcal{L}^{-1}\left(\frac{1}{s+1}\cdot\frac{1}{s-1}\cdot\frac{1}{s+5}\right)=\frac{1}{6}\sinh t+\frac{1}{24}(e^{-5t}-e^{-t})$$

$$\mathcal{L}^{-1}\left(\frac{s}{(s-1)(s+5)}\right)=\frac{1}{1-(-5)}(1\cdot e^{t}-(-5)e^{-5t})$$

Da <u>$\mathcal{L}$ er lineær</u> fås i alt, da $\sinh t=\frac{e^t-e^{-t}}{2}$

$$y(t)=\mathcal{L}^{-1}(Y)$$
$$=\frac{1}{6}\sinh t+\frac{1}{24}(e^{-5t}-e^{-t})-\frac{1}{6}(e^{t}+5e^{-5t})$$
$$=\frac{e^t}{12}-\frac{e^{-t}}{12}+\frac{e^{-5t}}{24}-\frac{e^{-t}}{24}-\frac{e^t}{6}-\frac{5}{6}e^{-5t}$$

så

$$\underline{\underline{y(t)=-\frac{1}{12}e^{t}-\frac{1}{8}e^{-t}-\frac{19}{24}e^{-5t}}}$$

OPG 1.2

$$e^{at} * e^{bt} = \int_0^t e^{a\cdot u} \cdot e^{b(t-u)}\, du \quad (\text{pr. def.})$$

$$= \int_0^t e^{bt+(a-b)u}\, du$$

$(a \neq b)$

$$= \left[\frac{1}{a-b} e^{bt+(a-b)u} \right]_{u=0}^{u=t}$$

$$= \frac{1}{a-b}\left(e^{bt+(a-b)t} - e^{bt+(a-b)\cdot 0} \right)$$

$$= \frac{e^{at} - e^{bt}}{a-b}$$

som ønsket.

Den generelle regel for foldninger giver

$$\mathcal{L}(e^{at} * e^{bt}) = \mathcal{L}(e^{at}) \cdot \mathcal{L}(e^{bt})$$

$$= \underline{\underline{\frac{1}{s-a} \cdot \frac{1}{s-b}}}$$

OPG 1.3: Vha. opg. 1.2 fås at

$$\frac{1}{s-1} \cdot \frac{1}{s+5} \cdot \frac{1}{s+1} = \mathcal{L}(e^{t} * e^{-5t}) \cdot \frac{1}{s+1}$$

$$= \mathcal{L}\left(\frac{e^{t} - e^{-5t}}{1-(-5)} \right) \cdot \mathcal{L}(e^{-t})$$

($\mathcal{L}$ lineær) $$= \frac{1}{6} \mathcal{L}\left((e^{t} - e^{-5t}) * e^{-t} \right)$$

$$= \frac{1}{6} \mathcal{L}\left(e^{t} * e^{-t} - e^{-5t} * e^{-t} \right)$$

$$= \frac{1}{6} \mathcal{L}\left(\frac{e^{t} - e^{-t}}{1-(-1)} - \frac{e^{-5t} - e^{-t}}{-5-(-1)} \right)$$

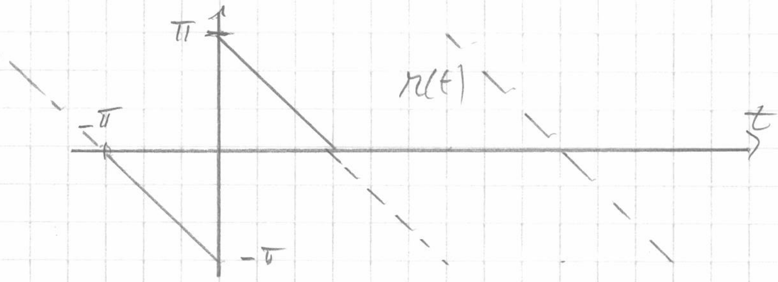
$$= \frac{1}{6} \mathcal{L}\left(\sinh(t) + \frac{1}{4}(e^{-5t} - e^{-t}) \right)$$

($\mathcal{L}$ lineær) $$= \underline{\underline{\mathcal{L}\left(\frac{1}{6}\sinh(t) + \frac{1}{24}(e^{-5t} - e^{-t}) \right)}}$$

Som ønsket

[NB! $\mathcal{L}\left(\frac{1}{6}\sinh(t) + \frac{1}{24}(e^{-5t} - e^{-t})\right)$ kan også bestemmes direkte, uden brug af foldninger.]

OPG 2: $r(t)$ ulige hvis
$$r(-t) = -r(t)$$



For $0 \le t \le \pi$:

$$r(-t) = -\pi-(-t) = -\pi+t = -(\pi-t) = -r(t)$$

og for $-\pi < t < 0$:

$$r(-t) = \pi-(-t) = \pi+t = -(-\pi-t) = -r(t).$$

Generelt: Hvert $s \in \mathbb{R}$ kan skrives
$$s = t + 2\pi \cdot p \text{ for } -\pi < t \le \pi,$$
p heltal

så

r 2π-periodisk

$$\underline{r(-s)} = r(-t-2\pi\cdot p)$$
$$= r(-t) = -r(t) = -r(t+2\pi\cdot p) = \underline{-r(s)}.$$

OPG 2.2: $r(t)$'s Fourierrække har kun sinus-led, da $r(t)$ ulige:
$$r(t) = \sum_{n=1}^{\infty} b_n \sin(nt)$$
med
$$b_n = \frac{2}{\pi}\int_0^{\pi} r(t)\sin(nt)\,dt$$
$$= \frac{2}{\pi}\int_0^{\pi}(\pi-t)\sin(nt)\,dt$$
$$= \frac{2}{\pi}\left[(\pi-t)\left(-\frac{\cos(nt)}{n}\right)\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{d}{dt}(\pi-t)\left(-\frac{1}{n}\cos(nt)\right)dt$$
$$= \frac{2}{\pi}\left(0+\pi\cdot\frac{\cos(0)}{n}\right) + \underbrace{\frac{2}{\pi}\left[-\frac{1}{n^2}\sin(nt)\right]_0^{\pi}}_{=0}$$
$$= \frac{2}{n} \qquad \text{da } \sin(n\cdot\pi) = 0.$$

Ialt
$$r(t) = \underline{\underline{\sum_{n=1}^{\infty} \frac{2}{n}\sin(nt)}}$$

OPG 2.3 Til $y''(t) - 3y'(t) + 28y(t) = r(t)$

gættes på

$$y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos(nt) + B_n \sin(nt) \right).$$

Med

$$y'(t) = \sum_{n=1}^{\infty} \left(-nA_n \sin(nt) + nB_n \cos(nt) \right)$$

$$y''(t) = \sum_{n=1}^{\infty} \left(-n^2 A_n \cos(nt) - n^2 B_n \sin(nt) \right)$$

fås

$$14A_0 + \sum_{n=1}^{\infty} \Big((-n^2 A_n - 3nB_n + 28A_n) \cos(nt) + (-n^2 B_n + 3nA_n + 28B_n) \sin(nt) \Big)$$

$$= 0 + \sum_{n=1}^{\infty} \left(0 \cdot \cos(nt) + \frac{2}{n} \sin(nt) \right)$$

Da Fourier-koefficienterne er nødt til at være ens, så er $A_0 = 0$ og

$$(-n^2 + 28) A_n - 3n B_n = 0$$

$$3n A_n + (28 - n^2) B_n = \frac{2}{n}$$

Her er $A_n = \frac{3n}{28 - n^2} B_n$

som giver

$$\left(\frac{(3n)^2}{28 - n^2} + 28 - n^2 \right) B_n = \frac{2}{n}$$

dvs

$$B_n = \frac{2/n}{\frac{9n^2}{28-n^2} + 28 - n^2} \cdot \frac{28 - n^2}{28 - n^2}$$

$$= \underline{\frac{\frac{2}{n}(28 - n^2)}{9n^2 + (28 - n^2)^2}} \leadsto \underline{A_n = \frac{6}{9n^2 + (28 - n^2)^2}}$$

Facit:

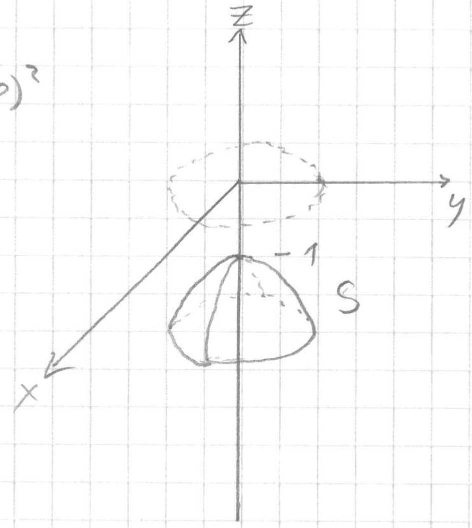
$$\underline{\underline{y(t) = \sum_{n=1}^{\infty} \left(\frac{6}{9n^2 + (28 - n^2)^2} \cos(nt) + \frac{2(28 - n^2)/n}{9n^2 + (28 - n^2)^2} \sin(nt) \right)}}$$

OPG 3 Når $(x, y, z) \in S$ så er

$$x = \rho\cos\varphi$$
$$y = 2\rho\sin\varphi$$
$$z = -1-(\rho\cos\varphi)^2 - (2\rho\sin\varphi)^2$$

så

$$\underline{\underline{z = -1-x^2-y^2}}$$



(S er den del af paraboloiden $z=-1-x^2-y^2$, der er under ellipsen $\left(\frac{x}{1}\right)^2 + \left(\frac{y}{1/\sqrt{2}}\right)^2 = 1$)

3.2

$$\vec{F}(x,y,z) = (2xy^3\cos^2 z,\ 3x^2y^2\cos^2 z,\ -x^2y^3\sin(2z))$$

Så

$$\nabla\times\vec{F} = \Big(\frac{\partial}{\partial y}(-x^2y^3\sin(2z)) - \frac{\partial}{\partial z}(3x^2y^2\cos^2 z),$$
$$\frac{\partial}{\partial z}(2xy^3\cos^2 z) - \frac{\partial}{\partial x}(x^2y^3\sin(2z)),$$
$$\frac{\partial}{\partial x}(3x^2y^2\cos^2 z) - \frac{\partial}{\partial y}(2xy^3\cos^2 z)\Big)$$

$$= \big(-3x^2y^2(\sin(2z) - 2\cos z\cdot\sin z),$$
$$2xy^3(2\cos z\cdot\sin z - \sin(2z)),$$
$$3xy^2(\cos^2 z - \cos^2 z)\ \big)$$

Pga. dobbeltvinkelformulen $\sin(2t) = 2\cos t\sin t$ giver dette:

(*) $$\nabla\times\vec{F} = (0,0,0).$$

Derfor er $\vec{F} = \nabla g$

(Kan også ses direkte at potentialet er $g(x,y,z) = x^2y^3\cos^2 z$.)

Da C er en lukket kurve fås heraf:

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \nabla g(r(\varphi))\cdot r'(\varphi)\,d\varphi = g(r(2\pi)) - g(r(0)) = \underline{\underline{0}}$$

3.3: Stokes: $\int_S (\nabla\times F)\cdot n\,dA = \int_C \vec{F}\cdot d\vec{r} = \underline{0}$ (jvf. (*))

OPG 4.1 Rækken er en kvotientrække med

$$q = -\frac{z^2}{7}$$

dvs

$$\sum_{n=0}^{\infty} \left(-\frac{1}{7}\right)^n z^{2n} = \sum_{n=0}^{\infty} \left(-\frac{z^2}{7}\right)^n = \sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$$

$$= \frac{1}{1-z^2/7} \cdot \frac{7}{7} \qquad = 7 \cdot f(z).$$

med konvergens når

$|q| < 1$ dvs. når $\left|\frac{z^2}{-7}\right| < 1$ el. $|z| < \sqrt{7}$

4.2. Fra opg. 4.1:

$$f(z) = \frac{1}{7} \cdot \sum_{n=0}^{\infty} \left(-\frac{1}{7}\right)^n z^{2n} \qquad = \sum_{n=0}^{\infty} \frac{(-1)^n}{7^{n+1}} z^{2n}$$

Konvergensradius R opfylder

$$R \geq \sqrt{7} \qquad (\text{pga. opg. } 4.1)$$

Men rækken divergerer for $q = 1$ $\left(\sum_{n=0}^{\infty} 1 = +\infty\right)$

dvs. for $z = i\sqrt{7}$, så

$$R \leq |i\sqrt{7}| = \sqrt{7}.$$

Ialt: $R = \sqrt{7}$

4.3 Da $$f(z) = \frac{1}{7+z^2} = \frac{1}{(z-i\sqrt{7})^1 (z+i\sqrt{7})^1}$$

Så har f poler af 1. orden i $z = \pm i\sqrt{7}$

$\frac{1}{z+i\sqrt{7}}$ er analytisk nær $z_1 = i\sqrt{7}$, med potensrække bestemt vha. $w = \frac{z-z_1}{i2\sqrt{7}}$:

$$\frac{1}{z+i\sqrt{7}} = \frac{1}{z-z_1+i2\sqrt{7}} = \frac{1}{i2\sqrt{7}} \cdot \frac{1}{1+\frac{z-z_1}{i2\sqrt{7}}} = \frac{-i}{2\sqrt{7}} \cdot \frac{1}{1-(-w)}$$

Dette giver

$$\frac{1}{z+i\sqrt{7}} = \frac{-i}{2\sqrt{7}} \cdot \sum_{n=0}^{\infty} (-w)^n \qquad (\text{når } |w|<1)$$

$$= \frac{-i}{2\sqrt{7}} \sum_{n=0}^{\infty} \frac{(-1)^n (z-z_1)^n}{(i\,2\sqrt{7})^n}$$

$$= -\frac{i}{2\sqrt{7}} \left(1 - \frac{z-z_1}{i2\sqrt{7}} + \dots + \frac{i^n}{(2\sqrt{7})^n}(z-z_1)^n + \dots \right)$$

så

$$f(z) = \frac{1}{z-z_1} \cdot \frac{-i}{2\sqrt{7}} \left(1 + \dots + \frac{i^n}{(2\sqrt{7})^n}(z-z_1)^n + \dots \right)$$

$$= \frac{-i}{2\sqrt{7}} \frac{1}{z-z_1} + \sum_{n=1}^{\infty} \frac{-i^{n+1}}{(2\sqrt{7})^{n+1}} (z-z_1)^{n-1}$$

Da potenserne $n-1$ løber i $\{0,1,2,\dots\}$ fås ved at erstatte $n-1$ med n:

$$\underline{\underline{f(z) = \frac{-i}{2\sqrt{7}} \frac{1}{z-z_1} + \sum_{n=1}^{\infty} \frac{i^n}{(2\sqrt{7})^{n+2}} (z-z_1)^n}}$$

4.4 Da $\frac{1}{7+x^2}$ har grad 2 i nævneren (og nul i tælleren) og ingen reelle rødder i nævneren:

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \operatorname{Res} f$$

med sum over den ene pol $z = z_1 = i\sqrt{7}$ i øvre halvplan.

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{-i}{2\sqrt{7}} = \text{koefficienten til } \frac{1}{z-z_1} \text{ i } 4.3$$

eller

$$\operatorname*{Res}_{z=z_1} f(z) = \frac{p(z_1)}{q'(z_1)} = \frac{1}{2z_1} = \frac{1}{2i\sqrt{7}} = \frac{-i}{2\sqrt{7}}$$

fås

$$\int_{-\infty}^{\infty} \frac{1}{7+x^2}\,dx = 2\pi i \cdot \frac{-i}{2\sqrt{7}} = \underline{\underline{\frac{\pi}{\sqrt{7}}}}$$

Som ønsket.