MAT3, prøve 11/08/2015

Opg 1: (a) Standardformlen (s. 214):

$$Y(s) = \frac{(s+a)y(0)+y'(0)}{s^2+as+b} + \frac{1}{s^2+as+b}\cdot R(s)$$

giver

$$Y(s) = \frac{(s+6)\cdot 1+3}{s^2+6s+13} + \frac{1}{s^2+6s+13}\cdot\frac{s+3}{(s+3)^2+2^2}$$

(b) Vi har
$$\frac{s+6+3}{s^2+6s+13} = \frac{(s+3)+6}{(s+3)^2+4} = \frac{s+3}{(s+3)^2+2^2} + 3\,\frac{2}{(s+3)^2+2^2}$$

så Tabel 6.1 giver

$$\mathcal{L}^{-1}\left(\frac{s+6+3}{s^2+6s+13}\right) = e^{-3t}\cos(2t) + 3e^{-3t}\sin(2t)$$

(c) $\mathcal{L}^{-1}$ er lineær, så vi behandler sidste led fra (a): Idet $s^2+6s+13 = (s+3)^2+4$ fås

$$\frac{1}{s^2+6s+13}\cdot\frac{s+3}{(s+3)^2+2^2} = \frac{s+3}{((s+3)^2+2^2)^2} \quad \leftarrow NB!$$

Da $\mathcal{L}\left(\frac{t}{2\omega}\sin(\omega t)\right) = \frac{s}{(s^2+\omega^2)^2}$ (Tabel 6.9, 22)

fås $\mathcal{L}\left(e^{-3t}\frac{t}{4}\sin(\omega t)\right) = \frac{s+3}{((s+3)^2+2^2)^2}$ (Forskydningssætningen)

Ialt:

$$y(t) = e^{-3t}\cos(2t) + 3e^{-3t}\sin(2t) + \frac{1}{4}te^{-3t}\sin(2t)$$

dvs.

$$y(t) = e^{-3t}\cos(2t) + \left(3+\frac{t}{4}\right)e^{-3t}\sin(2t)$$

Opg 2:

(a) $$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cosh(3t)e^{-int}\,dt$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi}\left(e^{(3-in)t} + e^{-(3+in)t}\right)dt$$

$$= \frac{1}{4\pi}\left[\frac{e^{(3-in)t}}{3-in} - \frac{e^{-(3+in)t}}{3+in}\right]_{-\pi}^{\pi} \qquad (3\pm in \neq 0)$$

$e^{\pm in\pi} = (-1)^n$
$$= \frac{1}{4\pi}\left[\frac{e^{3\pi}\cdot(-1)^n - e^{-3\pi}(-1)^n}{3-in} - \frac{e^{-3\pi}(-1)^n - e^{3\pi}(-1)^n}{3+in}\right]$$

$$= \frac{(-1)^n}{4\pi}\left(\frac{2\sinh(3\pi)}{3-in} + \frac{2\sinh(3\pi)}{3+in}\right)$$

$$= \frac{(-1)^n}{2\pi}\sinh(3\pi)\,\frac{(3+in)+(3-in)}{3^2-(in)^2} = \frac{\sinh(3\pi)}{\pi}\,\frac{3(-1)^n}{9+n^2}$$

(b) $y(t)$ 2π-per. $\Rightarrow$ $y(t) = \sum_{n=-\infty}^{\infty} K_n e^{int}$.

Indsættes: $$\sum_{-\infty}^{\infty}(-n^2+7in-13)K_n e^{int} = \sum_{-\infty}^{\infty}(-1)^n\frac{3}{9+n^2}e^{int}$$

Fourier koefficienter entydigt bestemte,

NB! $-n^2-13+7in \neq 0$

$$K_n = \frac{3(-1)^n}{(9+n^2)(-n^2+7in-13)} \quad \text{for alle } n\in\mathbb{Z}.$$

(c) Generelt $a_n = K_n + K_{-n}$, $b_n = i(K_n - K_{-n})$ for $n\geq 1$.

$$a_n = \frac{3(-1)^n(-n^2-7in-13)+3(-1)^n(-n^2+7in-13)}{(9+n^2)((n^2+13)^2+49n^2)}$$

$$= \frac{6(-1)^{n+1}(n^2+13)}{(9+n^2)((n^2+13)^2+49n^2)} \qquad b_n \text{ tilsvarende}$$

$a_0 = K_0 = \frac{3}{9(-13)} = \frac{-1}{39}$

Sæt

$D_n = (9+n^2)((n^2+13)^2+49n^2)$. Så er

$$y(t) = \frac{-1}{39} + \sum_{n=0}^{\infty}\left(\frac{(-1)^{n+1}(6n^2+78)}{D_n}\cos(nt) + \frac{(-1)^n 42n}{D_n}\sin(nt)\right)$$

Hverken lige eller ulige, da "$a_n = 0$ for alle n" og "$b_n = 0$ for alle n" er falske udsagn.

Opg 3:

(a) $$\text{rot}\,\vec{F} = \begin{pmatrix} 4(x+y+z)^3 + 6zx \\ -4(x+y+z)^3 + 3z^2y \\ 3x^2 - 3z^2 - (-4x^2 + z^3) \end{pmatrix}$$

$$\text{div}\,\vec{F} = 4(x+y+z)^3 - 8xy$$

(b) Green's sætning: Da $z=0$ i xy-planet,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint \vec{k}\cdot \text{rot}\,\vec{F}\,dx\,dy = \iint 7x^2\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^1 7\cdot r^2\cos^2\theta \cdot r\,dr\,d\theta$$

$$= 7\left[\tfrac{1}{4}r^4\right]_0^1 \cdot \left[\tfrac{\theta}{2} + \tfrac{1}{4}\sin(2\theta)\right]_{\theta=0}^{\theta=2\pi}$$

$$= \tfrac{7}{4}\cdot\left(\tfrac{2\pi}{2} - 0 + \tfrac{1}{4}(0-0)\right)$$

$$= \frac{7\pi}{4}$$

(c) Thm. 9.9.2: $\text{div}(\text{rot}\,\vec{F}) \equiv 0$

Derfor fås af divergenssætningen, idet $\vec{n}$ peger ud ad enhedskuglen,

$$\iint_S (\text{rot}\,\vec{F})\vec{n}\,dA = \iiint_K \text{div}(\text{rot}\,\vec{F})\,dx\,dy\,dz$$

$$= \iiint_K 0\,dx\,dy\,dz = 0$$

hvorved K betegner enhedskuglen i R^3

Opg 4:

(a) $f_1(z) = 3\frac{\sin z}{z}$ har hævelig sing. i $z=0$ med

$$f_1(z) = 3\frac{1}{z}\cdot\left(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \dots\right)$$
$$= 3 - \frac{3}{3!}z^2 + \frac{3}{5!}z^4 - \dots$$

Konvergensradius R fås til ∞:

$$R = \lim_{n\to\infty} \frac{(n+1)!\;3}{3\,(n-1)!} = \lim_{n\to\infty} (n+1)\cdot n = \underline{\underline{\infty}}$$

(b) $f_4(z) = \frac{\sin(3z)}{z^4}$ har pol i $z=0$,

$$f_4(z) = z^{-4}\left(3z - \frac{3^3}{3!}z^3 + \frac{3^5}{5!}z^5 - \dots\right)$$

(*) $$\underline{\underline{= \frac{3}{z^3} - \frac{9}{2}\cdot\frac{1}{z} + \frac{81}{40}z - \dots}} \qquad \underline{\underline{m=3}}$$

(c) $f_4(z)$ er analytisk på $\mathbb{C}$ pånær i $z=0$, jvf. (b), hvor der er en pol af orden 3:

Da (*) konvergerer for $|z|>0$ giver definitionen af residuet (S. 720) som koefficienten til $\frac{1}{z}$ i (*) at

$$\text{Res}(f_4(z), 0) = -\frac{9}{2}$$

Residuesætningen (S. 723) giver så

$$\int_C \frac{\sin(3z)}{z^4}\,dz = 2\pi i \sum_{z_0} \text{Res}(f_4(z), z_0)$$
$$= 2\pi i\left(-\frac{9}{2}\right)$$
$$= \underline{\underline{-9\pi i}}$$