MAT3, eksamen 7/1/2016

OPG 1: Tabel 6.9.18 giver

(a) $$\mathcal{L}(e^{-t}\cos(7t)) = \frac{s-(-1)}{(s-(-1))^2+7^2} = \underline{\underline{\frac{s+1}{(s+1)^2+7^2}}}$$

Af Thm. 6.3.1 fås så at

$$\mathcal{L}(r) = \mathcal{L}(u(t-1)e^{-(t-1)}\cos(7(t-1))) = e^{-s}\frac{s+1}{(s+1)^2+7^2}$$

Billedligningen (jvf. s. 214) er derfor

$$\underline{\underline{(s^2+2s+50)Y(s) = (s+2)\cdot 2 - 9 + e^{-s}\cdot\frac{s+1}{(s+1)^2+7^2}}}$$

(b) Fra (a) ses at

$$Y(s) = \frac{2s-5}{s^2+2s+50} + e^{-s}\cdot\frac{s+1}{((s+1)^2+7^2)(s^2+2s+50)}$$

Her indgår $y(0)$ og $y'(0)$ i første brøk.

Fordi $(s+1)^2+7^2 = s^2+2s+50$ fås

$$\mathcal{L}^{-1}\left(\frac{2s-5}{s^2+2s+50}\right) = \mathcal{L}^{-1}\left(\frac{2(s+1)-7}{(s+1)^2+7^2}\right)$$

($\mathcal{L}^{-1}$ lineær) $$= 2\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+7^2}\right) - \mathcal{L}^{-1}\left(\frac{7}{(s+1)^2+7^2}\right)$$

(Tabel 6.1) $$= \underline{\underline{2e^{-t}\cos(7t) - e^{-t}\sin(7t)}}$$

(c) For 2. brøk i $Y(s)$ fås

$$\frac{s+1}{(s+1)^2+7^2}\cdot\frac{1}{s^2+2s+50} = \frac{s+1}{((s+1)^2+7^2)^2}$$

$$= \mathcal{L}\left(e^{-t}\cdot\frac{t}{2\cdot 7}\sin(7t)\right)$$

ved at kombinere Tabel 6.9.22 og Thm. 6.1.2.

Faktoren e^{-s} kommer fra translation, så Thm 6.3.1 giver

$$\underline{y(t) = e^{-t}\left(2\cos(7t) - \sin(7t) + u(t-1)\cdot\frac{e}{14}(t-1)\sin(7(t-1))\right)}$$

OPGAVE 2: (a) Her er
$$f(t) = \begin{cases} 0 & \text{for } -\pi \le t < 0 \\ e^{-3t} & \text{for } 0 \le t < \pi \end{cases}$$
Derfor fås
$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt = \frac{1}{2\pi}\int_0^{\pi} e^{-3t}e^{-int}dt + 0$$
$$= \frac{1}{2\pi}\int_0^{\pi} e^{-(3+in)t}dt = \frac{1}{2\pi}\left[\frac{e^{-(3+in)t}}{-(3+in)}\right]_{t=0}^{t=\pi}$$

idet $3+in \neq 0$ i nævneren. Dvs.
$$c_n = \frac{1}{2\pi}\cdot\frac{-1}{3+in}\cdot\left(e^{-3\pi}\cdot e^{-in\pi} - e^0\right)$$
$$= \underline{\underline{\frac{1}{(6+i2n)\pi}\left(1 - e^{-3\pi}(-1)^n\right)}} \quad \text{for alle } n.$$

(b) Ved indsættelse af $y(t) = \sum_{n=-\infty}^{\infty} K_n e^{int}$ fås
$$\sum_{n=-\infty}^{\infty} K_n (in)^2 e^{int} + \sum_{n=-\infty}^{\infty} 5K_n in e^{int} - \sum_{n=-\infty}^{\infty} 3K_n e^{int} = \sum_{n=-\infty}^{\infty} c_n e^{int}$$
Dvs.
$$\sum_{n=-\infty}^{\infty} K_n(-n^2+5in-3)e^{int} = \sum_{n=-\infty}^{\infty} c_n e^{int}$$

Så fordi Fourier-koefficienter er entydigt bestemte, da ses heraf at
$$K_n(-n^2-3+5in) = c_n \quad \text{for alle } n.$$

Nu er $-n^2-3 \neq 0$ for alle n, så
$$K_n = \frac{c_n}{-n^2-3+5in} \quad \text{for alle } n.$$
Idet $c_0 = 7$ mens $c_n = 1/(4+n^2)$ for $n \neq 0$, fås
$$\underline{\underline{y(t) = -\frac{7}{3} + \sum_{n\neq 0} \frac{1}{(4+n^2)\cdot(-n^2-3+5in)} e^{int}}}$$

(c) Vi har $A_0 = K_0 = -\frac{7}{3}$

og for $n \geq 1$:

$$A_n = K_n + K_{-n}$$

$$= \frac{1}{(4+n^2)(-n^2-3+5in)} + \frac{1}{(4+(-n)^2)(-(-n)^2-3-5in)}$$

$$= \frac{1}{4+n^2} \cdot \frac{-n^2-3-5in+(-n^2-3+5in)}{(n^2+3)^2-(5in)^2}$$

$$= \frac{-2n^2-6}{(4+n^2)((n^2+3)^2+25n^2)}$$

Tilsvarende fås at

$$B_n = i(K_n - K_{-n})$$

$$= \frac{i}{4+n^2} \cdot \frac{-n^2-3-5in-(-n^2-3+5in)}{(n^2+3)^2-(5in)^2}$$

$$= \frac{10n}{(4+n^2)((n^2+3)^2+25n^2)}$$

Dette giver i alt at

$$\underline{\underline{y(t) = -\frac{7}{3} + \sum_{n=1}^{\infty} \left(\frac{-2n^2-6}{(4+n^2)((n^2+3)^2+25n^2)} \cos(nt) + \frac{10n}{(4+n^2)((n^2+3)^2+25n^2)} \sin(nt) \right)}}$$

Da hverken $A_n = 0$ for alle $n \geq 0$ eller $B_n = 0$ for alle $n \geq 1$ gælder, så er $\underline{\underline{y(t) \text{ hverken ulige eller lige}}}$.

<u>Opgave 3</u>: $\vec{F} = \begin{pmatrix} 6x^2z^4 + 2xz^3 \\ 2yz^3 \\ 8(xz)^3 + 3(x^2+y^2)z^2 \end{pmatrix}$

(a)

$$\text{rot}\,\vec{F} = \begin{pmatrix} 6yz^2 - 6yz^2 \\ 6\cdot 4\cdot x^2\cdot z^3 + 2\cdot 3xz^2 - (8\cdot 3x^2z^3 + 3\cdot 2z^2x) \\ 0 - 0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}}$$

Fordi domænet $D = \mathbb{R}^3$ ikke har nogen <u>huller</u>, så medfører $\text{rot}\,\vec{F} = \vec{0}$ at $\vec{F} = \text{grad}\, f$ for en funktion $f(x,y,z)$ (s. 419).

(b) Hvis $f(x,y,z)$ opfylder $\text{grad}\, f = \vec{F}$, så er

$$\frac{\partial f}{\partial x} = F_1 = 6x^2z^4 + 2xz^3$$

hvoraf

$$f(x,y,z) = 2x^3z^4 + x^2z^3 + g(y,z)$$

Nu er

$$2yz^3 = F_2 = \frac{\partial}{\partial y}(2x^3z^4 + x^2z^3 + g) = \frac{\partial g}{\partial y}$$

så

$$g(y,z) = y^2z^3 + h(z)$$

Men

$$8x^3z^3 + 3x^2z^2 + 3y^2z^2 = F_3 = \frac{\partial f}{\partial z}$$
$$= 8x^3z^3 + 3x^2z^2 + \frac{\partial}{\partial z}(y^2z^3 + h(z))$$

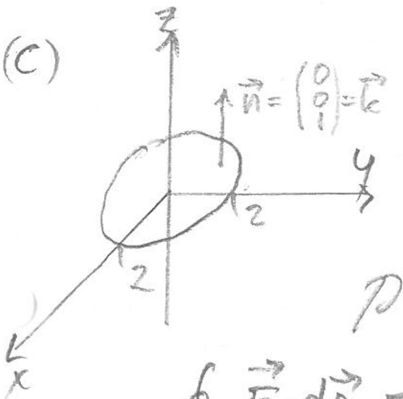
giver

$$0 = h'(z) \quad \text{, dvs } h(z) \text{ konstant}$$

Helt fås at $\vec{F}$ har potentialerne

$$\underline{\underline{f(x,y,z) = 2x^3z^4 + (x^2+y^2)z^3 + C, \quad C \text{ reel}}}$$

(c)



C er cirklen i xy-planen med radius 2 og centrum i (0, 0, 0). Stokes's sætning giver pga. punkt (a) at

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_{\odot} (\text{rot}\,\vec{F})\cdot\vec{k}\, dA = \iint \vec{0}\cdot\vec{k}\, dA = \underline{\underline{0}}.$$

Opgave 4:

(a) Rækken kan skrives

$$\sum_{n=1}^{\infty} \frac{2^n}{(6+8i)^n}(z+3+2i)^n = \sum_{n=1}^{\infty} \frac{1}{(3+4i)^n}(z-(-3-2i))^n$$

så centrum er i

$$\underline{\underline{z_0 = -3-2i}}$$

Desuden er

$$L^{**} = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} (|(3+4i)^{-n}|)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} (|3+4i|^{-n})^{\frac{1}{n}} = \lim_{n\to\infty} |3+4i|^{-1} = \frac{1}{5}$$

så konvergensradius $R = \frac{1}{L^{**}}$ er

$$\underline{\underline{R = 5}}$$

(b) $f(z) = \frac{1}{1-z} \cdot Ln(1-z)$ (begge for $|z|<1$)

(S. 695)
$$= (1+z+z^2+\ldots)\cdot\left((-z) - \frac{(-z)^2}{2} + \frac{(-z)^3}{3} - \frac{(-z)^4}{4} + \ldots\right)$$

$$= (1+z+z^2+\ldots)\cdot(-1)\cdot\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \ldots\right)$$

$$= -z - (1+\tfrac{1}{2})z^2 - (1+\tfrac{1}{2}+\tfrac{1}{3})z^3 - (1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4})z^4 - \ldots$$

$$= \underline{\underline{\sum_{n=1}^{\infty} -(1+\tfrac{1}{2}+\ldots+\tfrac{1}{n})z^n}} \quad (\text{for } |z|<1)$$

(c) med $z = x+iy$ er $g(z) = \exp(\bar{z}) = \exp(x-iy)$

$$= e^x\cos(-y) + ie^x\sin(-y)$$

(da cosinus er lige
sinus er ulige)

$$= \underbrace{e^x\cos y}_{=u} + i\underbrace{e^x(-\sin y)}_{=:v}$$

Så er

$$\frac{\partial u}{\partial x} = e^x\cos y \qquad \frac{\partial u}{\partial y} = -e^x\sin y$$

$$\frac{\partial v}{\partial x} = -e^x\sin y \qquad \frac{\partial v}{\partial y} = -e^x\cos y$$

Altså er hverken $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ eller $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ opfyldt, så ingen af Cauchy-Riemann er opfyldt. Dermed er $g(z) = e^{\bar{z}}$ <u>ikke analytisk ved noget punkt $z \in \mathbb{C}$.</u>