Reeksamen 31. august 2006

Opgave 1: (i) Af den karakteristiske ligning

$$0 = p_A(\lambda) = \lambda(\lambda+4)^3$$

ses at A har egenværdierne $\lambda = 0$ og $\lambda = -4$.
Da 0 har multiplicitet 1 og ligningen $(A - 0\cdot I)v = 0$ har en lineært uafhængig løsning (egenvektoren) er $\lambda = 0$ ej defekt.

Egenværdien -4 har multiplicitet 3. Og

$$(A+4I)v = 0 \Leftrightarrow \begin{bmatrix} +4 & 0 & 1 & 0 \\ 0 & +4 & 0 & 1 \\ -8 & 8 & -2 & 2 \\ 8 & -8 & 2 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(*) \qquad \Leftrightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ -4 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \\ -4 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

Dette viser at egenrummet E_{-4}, hørende til $\lambda = -4$, har dimension 2. Dermed er $\lambda = -4$ defekt med

$$d = 3 - 2 = 1.$$

(ii) For $\lambda = 0$ ses en egenvektor (direkte) at være $v_0 = (1, 1, 0, 0)$.
For $\lambda = -4$ haves fra $(*)$ to egenvektorer

$$v_1 = (1, 0, -4, 0), \qquad v_2 = (0, 1, 0, -4).$$

En generaliseret egenvektor fås af $(A+4I)^2 v = 0$ dvs

$$\begin{bmatrix} 8 & 8 & 2 & 2 \\ 8 & 8 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Her er $w_2 = (0, 0, 1, -1)$ en løsning, og

$$(A+4I)w_2 = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ -8 & 8 & -2 & 2 \\ 8 & -8 & 2 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -4 \\ 4 \end{bmatrix} =: w_1$$

Til $\lambda = -4$ haves derfor 2 kæder: (w_1, w_2) og (v_1).

For $\lambda = 0$ haves kæden (v_0) så løsningen er

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_0 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_1 e^{-4t} \begin{bmatrix} 1 \\ 0 \\ -4 \\ 0 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 1 \\ -1 \\ -4 \\ 4 \end{bmatrix}$$

$$+ c_3 e^{-4t} \left(t \cdot \begin{bmatrix} 1 \\ -1 \\ -4 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right), \quad c_0, c_1, c_2, c_3 \in \mathbb{R}$$

gave 2 (i) Differentieres vha. kædereglen fås

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(F(x+ct) + G(x-ct)\right) = F'(x+ct) \cdot c - cG'(x-ct)$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(F'(x+ct) \cdot c - cG'(x-ct)\right) = F''(x+ct) \cdot c^2 + c^2 G''(x-ct).$$

og $$\frac{\partial^2 u}{\partial x^2} = F''(x+ct) \cdot 1^2 + 1^2 G''(x-ct)$$

Ved sammenligning ses at $\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}$, som ønsket

Hvis $F(x) + G(x) = g(x)$ og $cF'(x) - cG'(x) = h(x)$, så er

$$u(x,0) = F(x + c \cdot 0) + G(x - c \cdot 0) = g(x)$$
$$u_t(x,0) = cF'(x + c \cdot 0) - cG'(x - c \cdot 0) = h(x) \quad \text{(jvf. (4))}$$

Så $u(x,t)$ opfylder også begyndelses-betingelserne når (4) gælder.

(ii) Integreres identiteten $cF'(x) - cG'(x) = h(x)$ fås

$$\int_0^x h(s)\,ds = \int_0^x \left(cF'(s) - cG'(s)\right) ds = cF(x) - cG(x) - \left(cF(0) - cG(0)\right).$$

Størrelsen i parentesen er en vis konstant, der betegnes med k.

For hvert x løser tallene $F(x)$, $G(x)$ de to ligninger

$$\begin{cases} F(x) + G(x) = g(x) \\ cF(x) - cG(x) = \int_0^x h(s)\,ds + k \end{cases}$$

Så er $F(x) - G(x) = \frac{1}{c}\int_0^x h(s)\,ds + \frac{1}{c}k$ og adderes $F+G$ fås

$$F(x) = \tfrac{1}{2}g(x) + \tfrac{1}{2c}\int_0^x h(s)\,ds + \tfrac{1}{2c}k .$$

Og

$$G(x) = g(x) - F(x) = \tfrac{1}{2}g(x) - \tfrac{1}{2c}\int_0^x h(s)\,ds - \tfrac{1}{2c}k .$$

Indsætter $x \pm ct$ fås af (3) at

$$u(x,t) = \tfrac{1}{2}g(x+ct) + \tfrac{1}{2c}\int_0^{x+ct} h(s)\,ds + \tfrac{1}{2c}k$$
$$+ \tfrac{1}{2}g(x-ct) - \tfrac{1}{2c}\int_0^{x-ct} h(s)\,ds - \tfrac{1}{2c}k$$

$$= \tfrac{1}{2}\left(g(x+ct) + g(x-ct)\right) + \tfrac{1}{2c}\int_{x-ct}^{x+ct} h(s)\,ds .$$

Som ønsket.

Opgave 3. (i) Den fysiske betydning af randbetingelserne i (5) er at stangen holdes isoleret på endeflader ved $x=0$ og $x=L$. (Neumann-betingelsen)

(ii) Pga. Neumann-betingelsen haves formlen

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\left(\frac{n\pi}{L}\right)^2 kt} \cos\left(\frac{n\pi}{L}\cdot x\right),$$

hvor

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}\cdot x\right)dx , \quad n = 0,1,2,3,\dots$$

For $n=0$:

$$a_0 = \frac{2}{L}\int_0^L (L-x)\,dx = \frac{1}{L}\left[-(L-x)^2\right]_0^L = L.$$

$n \geq 1$:

$$a_n = \frac{2}{L}\int_0^L (L-x)\cos\left(\frac{n\pi}{L}x\right)dx$$

$$= \frac{2}{L}\left[(L-x)\frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\right]_0^L - \frac{2}{L}\int_0^L \frac{-L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)dx$$

$$= 0 - \frac{2}{L}\left[\left(\frac{L}{n\pi}\right)^2\cos\left(\frac{n\pi}{L}x\right)\right]_0^L$$

$$= -\frac{2L}{n^2\pi^2}\left(\cos(n\cdot\pi)-1\right) = \begin{cases} 0, & \text{for } n \text{ lige} \\ \frac{4L}{n^2\pi^2}, & \text{for ulige } n. \end{cases}$$

Deraf fås

$$\underline{\underline{u(x,t) = \frac{L}{2} + \sum_{n \text{ ulige}} \frac{4L}{n^2\pi^2} e^{-\left(\frac{n\pi}{L}\right)^2 k t}\cos\left(\frac{n\pi}{L}x\right).}}$$

ii)

For $w(x) = W\cdot x$ er det klart at $\frac{\partial w}{\partial t} = 0$ og $\frac{\partial^2 w}{\partial x^2} = 0$, så $w_t - k w_{xx} = 0$, som ønsket. Og da $w_x(x) = W$ (konstant) ses at

$$W_x(0) = W, \quad W_x(L) = W$$

For $u(x,t) = v(x,t) - w(x)$ fås

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(v-w) = \frac{\partial v}{\partial t} - \frac{\partial w}{\partial t} = \frac{\partial v}{\partial t}$$

$$k\frac{\partial^2 u}{\partial x^2} = k\frac{\partial^2}{\partial x^2}(v-w) = k\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 w}{\partial x^2}\right) = k\frac{\partial^2 v}{\partial x^2}$$

Da højresiderne er ens (fordi v løser (6)) er også venstresiderne ens, dvs

$$\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0.$$

Tilsvarende fås

$$u(x,0) = v(x,0) - w(x) = g(x) - w(x) \text{, som ønsket}$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial v}{\partial x}(0,t) - W = W - W = 0$$

$$\frac{\partial u}{\partial x}(L,t) = \frac{\partial v}{\partial x}(L,t) - W = W - W = 0$$

(iv) Da $u(x,t)$ løser det homogene Neumann problem (5) for

$$f(x) = g(x) - w(x) = LW - Wx = W(L-x)$$

får man løsningen fra (ii) ganget med W, dvs

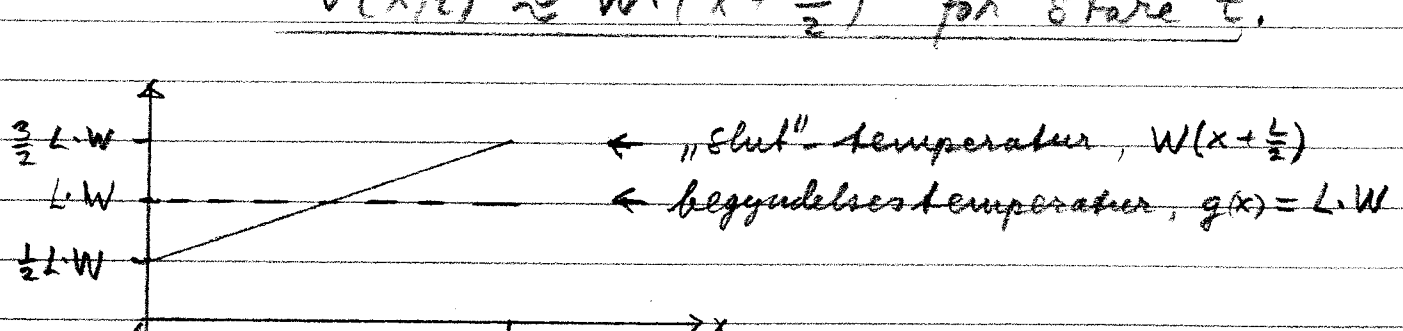
$$u(x,t) = \tfrac{1}{2}LW + \sum_{n=1,3,5,\dots} \frac{4LW}{n^2\pi^2} e^{-\frac{n^2\pi^2}{L^2}kt} \cos\left(\frac{n\pi}{L}x\right)$$

Idet $v(x,t) = u(x,t) + w(x)$ fås

$$\underline{\underline{v(x,t) = W\left(x + \frac{L}{2}\right) + \sum_{n \text{ ulige}} \frac{4LW}{n^2\pi^2} e^{-\left(\frac{n\pi}{L}\right)^2 kt} \cos\left(\frac{n\pi}{L}x\right)}}$$

Da $-kt \to -\infty$ for $t \to \infty$, går hvert led i $\sum_{n \text{ ulige}}$ mod 0 for $t \to \infty$, så

$$\underline{v(x,t) \approx W \cdot \left(x + \frac{L}{2}\right) \text{ for store } t.}$$



<u>(Op)gave 4</u> $B-5I = \begin{bmatrix} 0 & 0 & 0 \\ 10 & 0 & 0 \\ 2 & 30 & 0 \end{bmatrix} =: C$

giver

$$C^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 300 & 0 & 0 \end{bmatrix}, \quad C^3 = 0.$$

Dermed er, da $C\cdot(5I) = (5I)\cdot C$,

$$e^{tB} = e^{t(5I+C)}$$

$$= e^{t5I}\cdot e^{tC} = (e^{5t}\cdot I)(I + tC + \tfrac{1}{2}(tC)^2)$$

$$= e^{5t}\begin{pmatrix} 1 & 0 & 0 \\ 10t & 1 & 0 \\ 150t^2+2t & 30t & 1 \end{pmatrix}.$$

Den homogene ligning + begyndelsesbetingelse løses af

$$x_h(t) = e^{tB}\cdot x(0) = e^{5t}\begin{pmatrix} 4 \\ 40t+5 \\ 600t^2+158t+6 \end{pmatrix}$$

$$= e^{5t}\left(\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + t\begin{pmatrix} 0 \\ 40 \\ 158 \end{pmatrix} + t^2\begin{pmatrix} 0 \\ 0 \\ 600 \end{pmatrix}\right)$$

(ii) Problemet (7) løses af

$$x(t) = x_h(t) + \int_0^t e^{(t-s)B} f(s)\,ds$$

$$= x_h(t) + \int_0^t e^{5(t-s)}\begin{pmatrix} 1 & 0 & 0 \\ 10(t-s) & 1 & 0 \\ 150(t-s)^2+2(t-s) & 30(t-s) & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ds$$

$$= x_h(t) + \int_0^t e^{5(t-s)}\begin{pmatrix} 0 \\ 1 \\ 30(t-s) \end{pmatrix} ds$$

Her er $\int_0^t e^{5(t-s)}\,ds = \left[-\frac{1}{5}e^{5(t-s)}\right]_0^t = \frac{1}{5}e^{5t} - \frac{1}{5}$, mens

$$\int_0^t e^{5(t-s)}30(t-s)\,ds = \left[-\tfrac{1}{5}e^{5(t-s)}\cdot 30(t-s)\right]_0^t - \int_0^t \tfrac{1}{5}e^{5(t-s)}\cdot 30\,ds$$
$$= 6te^{5t} - 6\left(\tfrac{1}{5}e^{5t} - \tfrac{1}{5}\right)$$

Det giver

$$x(t) = x_h(t) + te^{5t}\begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} + \frac{1}{5}e^{5t}\begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 0 \\ -1 \\ 6 \end{bmatrix}$$