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Solution to exercize Kreyszig 6.3.23

NB:I will not show the derivation how to find the coefficients involved in the partial fraction decompositions, for a clear example of this method see the solution of exercize 6.7.19.

$$y'' + ay' + by = r(t) \quad ; \quad y(0) = 0\,;\, y'(0) = -1$$

$$\begin{aligned} a &= 1\,;\, b = -2 \\ r(t) &= (3\sin(t) - \cos(t))[1 - u(t-2\pi)] + (3\sin(2t) - \cos(2t)u(t-2\pi) \\ &= (3\sin(t) - \cos(t)) + (\cos(t) - 3\sin(t) + 3sin(2t) - \cos(2t)u(t-2\pi) \end{aligned}$$

By table:

$$\begin{aligned} R(s) &= \frac{3}{s^2+1} - \frac{s}{s^2+1} + \left[\frac{s}{s^2+1} - \frac{3}{s^2+1} + 3\frac{2}{s^2+4} - \frac{s}{s^2+4}\right] e^{-2\pi s} \\ &= \frac{3-s}{s^2+1} + \left[\frac{s-3}{s^2+1} + \frac{6-s}{s^2+4}\right] e^{-2\pi s} \end{aligned}$$

General form of the solution is:

$$(s^2 + as + b)Y(s) = (s+a)y(0) + y'(0) + R(s)$$

So in this case:

$$\begin{aligned} (s^2 + s + -2)Y(s) &= -1 + R(s) \\ (s+2)(s-1)Y(s) &= -1 + (1 - e^{-2\pi s})\frac{3-s}{s^2+1} + \frac{6-s}{s^2+4}e^{-2\pi s} \\ &= I + II + III \end{aligned}$$

Solve term for term, to keep things clear:

Term I:

$$(s+2)(s-1)Y(s) = -1$$

$$\begin{aligned} Y(s) &= \frac{-1}{(s+2)(s-1)} \\ Y(s) &= A/(s+2) + B/(s-1) \quad \text{By solving the PFD: } A = 1/3;\ B = -1/3 \\ Y(s) &= (1/3)/(s+2) - (1/3)/(s-1) \text{By solving the PFD:} A = 1/3;\ B = -1/3 \end{aligned}$$

By aid of the table

$$y_I(t) = (1/3)e^{-2t} - (1/3)e^t$$

Term II:

$$\begin{aligned}
(s+2)(s-1)Y(s) &= \frac{3-s}{s^2+1}(1-e^{-2\pi s}) \\
Y(s) &= \frac{3-s}{(s^2+1)(s+2)(s-1)}(1-e^{-2\pi s}) \\
Y(s) &= \frac{As+B}{s^2+1} + \frac{C}{s+2} + \frac{D}{s-1}(1-e^{-2\pi s})
\end{aligned}$$

By solving for the coefficients:

$A = 0$; $B = -1$; $C = -1/3$; $D = 1/3$

$$Y(s) = \frac{-1}{s^2+1} + \frac{-1/3}{s+2} + \frac{1/3}{s-1}(1-e^{-2\pi s})$$

By aid of table and 2nd shifting theorem (page 219)

$$y_{II}(t) = -\sin(t) - (1/3)e^{-2t} + (1/3)e^{t} - (-\sin(t-2\pi) - (1/3)e^{-2(t-2\pi} + (1/3)e^{t-2\pi})u(t-2\pi))$$

Term III:

$$\begin{aligned}
(s+2)(s-1)Y(s) &= \frac{6-s}{s^2+4}e^{-2\pi s} \\
Y(s) &= \frac{6-s}{(s^2+4)(s+2)(s-1)}e^{-2\pi s} \\
vY(s) &= \frac{As+B}{s^2+4}\frac{C}{s+2}\frac{D}{s-1}e^{-2\pi s}
\end{aligned}$$

By solving for the coefficients:

$A = 0$; $B = -1$; $C = -1/3$; $D = 1/3$

$$Y(s) = \frac{-1}{s^2+4}\frac{-1/3}{s+2}\frac{1/3}{s-1}e^{-2\pi s}$$

By aid of table and 2nd shifting theorem (page 219)

$$y_{III}(t) = [-(1/2)\sin 2(t-2\pi) - (1/3)e^{-2(t-2\pi)} + (1/3)e^{(t-2\pi)}]u(t-2\pi)$$

Summing the 3 terms gives

$$\begin{aligned}
y(t) &= y_I(t) + y_{II}(t) + y_{III}(t) \\
&= -\sin(t) + \sin(t-2\pi)u(t-2\pi)) - (1/2)\sin 2(t-2\pi)u(t-2\pi)
\end{aligned}$$

Which can be written as

$$y(t) = \begin{cases} -\sin(t) & ; \quad 0 < t < 2\pi \\ (-1/2)\sin(2t) & ; \quad t > 2\pi \end{cases}$$