8.5.3 $\quad u_t = 2u_{xx}$ for $0 < x < 1$, $0 < t$

$$u(0,t) = 0 = u(1,t)$$

$$u(x,0) = 5\sin(\pi x) - \frac{1}{5}\sin(3\pi x)$$

Data $\quad f(x) = 5\sin(\pi x) - \frac{1}{5}\sin(3\pi x)$

er (defineret for $x \in \mathbb{R}$ og er) en ulige funktion m. periode 1. Derfor sinus-række:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

Men tallene b_n er entydigt bestemte, så

$$b_n = \begin{cases} 5 & \text{for } n = 1 \\ 0 & \text{for } n = 2 \\ -1/5 & \text{for } n = 3 \\ 0 & \text{for } n \geq 4 \end{cases}$$

Dirichletproblemets løsningsformel giver så

$$u(x,t) = \sum_{n=1}^{\infty} b_n \exp\left(-n^2\frac{\pi^2}{1^2}2\cdot t\right)\sin\left(n\frac{\pi}{1}x\right)$$

$$= \underline{\underline{5e^{-2\pi^2 t}\sin(\pi x) - \frac{1}{5}e^{-18\pi^2 t}\sin(3\pi x)}}.$$

NB! Bemærk hvordan f's „højfrekvente" bidrag, $\sin(3\pi x)$, for $t>0$ bliver „dræbt" hurtigst, da $e^{-18\pi^2 t}$ er langt mindre end $e^{-2\pi^2 t}$!

(8.5.7) Neumann problem:
$$\begin{cases} u_t = \frac{1}{3} u_{xx}, & \text{for } 0<x<2, \quad 0<t \\ u_x(0,t) = 0 = u_x(2,t) & \text{for } t>0 \\ u(x,0) = \cos^2(2\pi x) \end{cases}$$

Almen løsningsformel:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp(-n^2 \tfrac{\pi^2}{L^2} kt) \cos(n \tfrac{\pi}{L} x)$$

hvor a_n for $n \geq 0$ er koefficienterne i data's cosinus-række:

$$f(x) = \cos^2(2\pi x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n \tfrac{\pi}{2} x)$$

Da $f(x)$ allerede er trigonometrisk, opløses den vha. Eulers formler $(\cos t = \frac{1}{2}(e^{it} + e^{-it}))$:

$$f(x) = \left[\tfrac{1}{2}(e^{i2\pi x} + e^{-i2\pi x})\right]^2 = \tfrac{1}{4}\left((e^{i2\pi x})^2 + (e^{-i2\pi x})^2 + 2e^{i2\pi x} \cdot e^{-i2\pi x}\right)$$

(1) $$= \tfrac{1}{4}(e^{i4\pi x} + e^{-i4\pi x} + 2) = \tfrac{1}{2} + \tfrac{1}{2}\cos(4\pi x)$$

Entydigheden af $a_0, a_1, a_2, \dots$ giver nu

(*) $$a_0 = 1, \quad a_1 = a_2 = \dots = a_7 = 0, \quad a_8 = \tfrac{1}{2}, \quad a_n = 0 \text{ for } n \geq 9$$

Endelig:

$$\underline{\underline{u(x,t) = \tfrac{1}{2} + \tfrac{1}{2} e^{-(16\pi^2/3)t} \cos(4\pi x)}}$$

(1) Dette kunne også opnås vha. dobbeltvinkelformlen

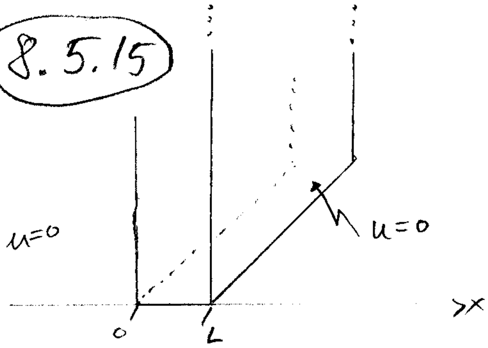
$$\cos(2t) = 2\cos^2 t - 1.$$

Eller man kunne udregne $a_n = \frac{2}{2}\int_0^2 \cos^2(2\pi x)\cos(n\frac{\pi}{2}x)\,dx$, hvilket er møjsommeligt.

Hvad giver din lommeregner om dette integral?

(giver den resultatet i (*) ovenfor?)

8.5.15



Dirichlet problem
$$\begin{cases} u_t = k u_{xx}, & \text{for } 0<x<L,\ t>0 \\ u(0,t) = 0 \\ \quad\ \ = u(L,t) & \text{for } t>0 \\ u(x,0) = f(x) \end{cases}$$

hvor data
$$f(x) = \begin{cases} A, & \text{for } 0<x<\frac{L}{2} \\ 0, & \text{for } \frac{L}{2} \le x \le L. \end{cases}$$

Sinusrækken
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(n\frac{\pi}{L}x\right)$$

har koefficienter

(jvf. s. 562)
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(n\frac{\pi}{L}x\right)dx$$
$$= \frac{2}{L}\int_0^{L/2} A\sin\left(n\frac{\pi}{L}x\right)dx$$
$$= \frac{2A}{L}\left[\frac{L}{n\pi}\cos\left(n\frac{\pi}{L}x\right)\right]_{x=0}^{L/2}$$
$$= -\frac{2A}{n\pi}\left(\cos\left(n\frac{\pi}{2}\right) - 1\right)$$

Dobbeltvinkelformlen $\cos(2t) = 2\cos^2 t - 1 = 1 - 2\sin^2 t$

giver
$$b_n = -\frac{2A}{n\pi}\left(\cos\left(n\frac{\pi}{2}\right) - 1\right)$$
$$= -\frac{2A}{n\pi}\left(-2\sin^2\left(n\frac{\pi}{4}\right)\right) = +\frac{4A}{n\pi}\sin^2\left(n\frac{\pi}{4}\right)$$

Via løsningsformlen fås

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4A}{n\pi}\sin^2\left(n\frac{\pi}{4}\right)\exp\left(-n^2\frac{\pi^2}{L^2}kt\right)\sin\left(n\frac{\pi}{L}x\right).$$

8.5.16 Fra opg. 15, med $A = 100$, $L = 50$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{400}{\pi \cdot n} \sin^2(n \cdot \tfrac{\pi}{4}) \cdot \sin(n \cdot \tfrac{\pi}{50} x) \exp(-n^2 \tfrac{\pi^2}{2500} k \cdot t)$$

$$u(25,1800) = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin^2(n \cdot \tfrac{\pi}{4}) \sin(n \cdot \tfrac{\pi}{2}) \exp(-n^2 \pi^2 \tfrac{18}{25} k)$$

Her er $\sin(n \cdot \frac{\pi}{2}) = 0$ for alle lige n.

Hvorimod $\sin^2(n \cdot \frac{\pi}{4}) = (\pm \frac{\sqrt{2}}{2})^2 = \frac{1}{2}$ for alle ulige n.

Indføres $n = 2m-1$ for $m \geq 1$ fås

$$u(25,1800) = \frac{200}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2m-1} \exp(-(2m-1)^2 \pi^2 \tfrac{18}{25} k).$$

(a) jern: $k = 0{,}15$, så $18k/25 = 0{,}108$ og $18\pi^2 k/25 \simeq 1{,}07$

$$u(25,1800) \simeq \frac{200}{\pi}(0{,}3444 - 0{,}000023 + \dots)$$

$$\simeq 22$$

(b) beton: $k = 5 \cdot 10^{-3}$.

Ønsker: $u(25,t) = 22$.

Tilnærmer: $\frac{200}{\pi} e^{-\pi^2 \frac{k}{2500} \cdot t} = 22$

$$-\pi^2 \frac{5}{25} \cdot 10^{-5} \cdot t = \ln(22 \cdot \pi/200) = \ln(\tfrac{11 \cdot \pi}{100})$$

$$t = -\frac{5 \cdot 10^5}{\pi^2} \ln(\tfrac{11 \cdot \pi}{100})$$

$$= 53829 \text{ s}.$$

$$= \underline{\underline{14{,}95 \text{ h}}}$$