Exam - Mathematics for Computer Graphics, Answers

Thursday, January 5, 2012, 9.00–13.00.

Exercise 1

1. (Use (4.8) on page 146.) We have

$$\theta_x = 90^\circ, \quad Cx = \cos\theta_x = 0, \quad Sx = \sin\theta_x = 1.$$

$$\theta_y = 180^\circ, \quad Cy = \cos\theta_y = -1, \quad Sy = \sin\theta_y = 0.$$

$$\theta_z = 90^\circ, \quad Cz = \cos\theta_z = 0, \quad Sz = \sin\theta_z = 1.$$

Using this we get

$$R = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

2. Then $R\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Exercise 2

The general way to find a matrix A so that $S(\mathbf{x}) = A\mathbf{x}$ where S is some linear transformation is the following: Compute

$$\mathbf{a}_0 = S(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix},$$

$$\mathbf{a}_1 = S(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}, \text{ and}$$

$$\mathbf{a}_2 = S(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}.$$

Then $A = [\mathbf{a}_0\ \mathbf{a}_1\ \mathbf{a}_2] = \begin{bmatrix} 0 & -3 & -1 \\ 3 & 0 & -2 \\ 1 & 2 & 0 \end{bmatrix}$.

For this particular linear transformation we may also use $A = \tilde{\mathbf{v}}$ on page 98 with $v_x = 2, v_y = -1, v_z = 3$.

Exercise 3

1. Let $\mathbf{u} = P_1 - P_0 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = P_2 - P_0 = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$.

We compute $\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{bmatrix} -5 \\ 5 \\ -10 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

The equation for the plane is then

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where $P_0 = (x_0, y_0, z_0) = (-1, 0, 1)$.

Inserting the numbers we get

$$-5(x + 1) + 5(y - 0) - 10(z - 1) = 0$$

or

$$-x + y - 2z + 1 = 0.$$

2. We see that $(x, y, z) = (0, 1, 1)$ is a solution to the equation. Thus $P = (0, 1, 1)$ is a point on the plane.

3. Let $\mathbf{w} = P - P_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ apply the method from page 84-85:

$\mathbf{v} \times \mathbf{w} = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} = s(\mathbf{v} \times \mathbf{u}) = s \begin{bmatrix} 5 \\ -5 \\ 10 \end{bmatrix}$. Thus $s = \frac{2}{5}$.

$\mathbf{u} \times \mathbf{w} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = t(\mathbf{u} \times \mathbf{v}) = t \begin{bmatrix} -5 \\ 5 \\ -10 \end{bmatrix}$. Thus $t = \frac{1}{5}$.

The barycentric coordinates are

$$(1 - s - t, s, t) = (\frac{2}{5}, \frac{2}{5}, \frac{1}{5}).$$

Exercise 4

1. (page 440) $Q(u) = UMG = [u^3 \; u^2 \; u \; 1] \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & -2 & 1 \\ -1 & 2 & 1 \end{bmatrix} =$

$u^3(0, -2, -2) + u^2(-2, 5, 3) + u(3, -2, 1) + (1, 0, 0)$.

2. For $u = \frac{1}{2}$ we get $Q(\frac{1}{2}) = (2, 0, 1)$.

Exercise 5

1. $\cos(\theta) = p{\cdot}q = 0 \cdot \frac{4}{5} + \frac{3}{5} \cdot 0 + 0 \cdot \frac{3}{5} - \frac{4}{5} \cdot 0 = 0$. Thus $\theta = 90°$. Since $\sin(\theta) = 1$ we get

$$\text{slerp}(p, q, t) = \sin((1-t)90°)p + \sin(t90°)q =$$

$$(\frac{4}{5}\sin(t90°), \frac{3}{5}\sin((1-t)90°), \frac{3}{5}\sin(t90°), -\frac{4}{5}\sin((1-t)90°)).$$

2. If $t = \frac{2}{3}$ then $\sin((1-t)90°) = \sin(30°) = \frac{1}{2}$ and $\sin(t90°) = \sin(60°) = \frac{\sqrt{3}}{2}$. Thus

$$\text{slerp}(p, q, \frac{2}{3}) = (\frac{2\sqrt{3}}{5}, \frac{3}{10}, \frac{3\sqrt{3}}{10}, -\frac{2}{5}).$$

Exercise 6

(We use (2.13), (2.14) and (2.15) on page 73.)
$x = \rho \sin\phi \ \cos\theta = 3\sin\frac{\pi}{4} \ \cos\frac{\pi}{2} = 3 \cdot \frac{\sqrt{2}}{2} \cdot 0 = 0.$
$y = \rho \sin\phi \ \sin\theta = 3\sin\frac{\pi}{4} \ \sin\frac{\pi}{2} = 3 \cdot \frac{\sqrt{2}}{2} \cdot 1 = \frac{3\sqrt{2}}{2}.$
$z = \rho\cos\phi = 3\cos\frac{\pi}{4} = 3 \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}.$

The Cartesian coordinates are $(x, y, z) = (0, \frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2})$.

Exercise 7

(page 186) $\mathbf{r} = \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$ has length $||\mathbf{r}|| = \sqrt{3^2 + (-6)^2 + 2^2} = \sqrt{49} = 7$. Thus $\hat{\mathbf{r}} = \frac{1}{7}\mathbf{r} = \begin{bmatrix} \frac{3}{7} \\ -\frac{6}{7} \\ \frac{2}{7} \end{bmatrix}$.

$\theta = 60°$. We compute $\cos(\frac{\theta}{2}) = \frac{\sqrt{3}}{2}$ and $\sin(\frac{\theta}{2}) = \frac{1}{2}$.
The quaternion representing this rotation is:

$$(\cos(\frac{\theta}{2}), \sin(\frac{\theta}{2})\hat{\mathbf{r}}) = (\frac{\sqrt{3}}{2}, \frac{1}{2}\begin{bmatrix} \frac{3}{7} \\ -\frac{6}{7} \\ \frac{2}{7} \end{bmatrix}) = (\frac{\sqrt{3}}{2}, \frac{3}{14}, -\frac{3}{7}, \frac{1}{7}).$$

Exercise 8

1. $SS^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$RR^T = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Since $SS^T = I$ and $RR^T = I$, S and R are both orthogonal.

2. $\det S = -1$ and $\det R = 1$. Thus R is a rotation matrix but S is not.

3. (page 182–183) $\text{trace}(R) = 0 + 0 + 0 = 0$. Then the rotation angle is $\theta = \arccos(\frac{\text{trace}(R)-1}{2}) = \arccos(-\frac{1}{2}) = 120°$.

We compute the rotation axis as follows:

$$\mathbf{r} = (R_{21}-R_{12}, R_{02}-R_{20}, R_{10}-R_{01})^T = (1-0, -1-0, -1-0)^T = (1, -1, -1)^T.$$

This vector has length $||\mathbf{r}|| = \sqrt{1^2 + (-1)^2 + (-1)^2} = \sqrt{3}$, and so

$$\hat{\mathbf{r}} = \frac{1}{\sqrt{3}}\mathbf{r} = (\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}})^T.$$