MCG - 3

$\mathbf{u}, \mathbf{v}, \mathbf{w}$: three linearly independent vectors in $\mathbb{R}^3$.

Use right hand:
index finger points in direction $\mathbf{u}$
middle finger points in direction $\mathbf{v}$.

Then we say that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is right-handed if $\mathbf{w}$ is on the same side of the plane spanned by $\mathbf{u}, \mathbf{v}$ as the thumb.

Otherwise $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is left-handed.

Example: $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is right-handed.

Let $\mathbf{v} = (v_x, v_y, v_z)$ and $\mathbf{w} = (w_x, w_y, w_z)$.

Then the cross product is defined by

$$\mathbf{v} \times \mathbf{w} = (v_y w_z - w_y v_z, v_z w_x - w_z v_x, v_x w_y - w_x v_y).$$

$\mathbf{v} \times \mathbf{w}$ is the vector orthogonal to $\mathbf{v}$ and $\mathbf{w}$, satisfying that:

$\mathbf{v}, \mathbf{w}, \mathbf{v} \times \mathbf{w}$ is right-handed and

$||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}|| \; ||\mathbf{w}|| \sin\theta$,

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

$||\mathbf{v} \times \mathbf{w}||$ is the area of a parallelogram where $\mathbf{v}$ and $\mathbf{w}$ are two edges.

Vector triple product:

If $\mathbf{v}$ and $\mathbf{w}$ are two vectors in $\mathbb{R}^3$ (non-parallel) then

$$\mathbf{w},\ \mathbf{v} \times \mathbf{w},\ \mathbf{w} \times (\mathbf{v} \times \mathbf{w})$$

is a right-handed orthogonal basis.
(Alternative to Gram-Schmidt.)

Scalar triple product:

$$\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = \mathbf{w}\cdot(\mathbf{u} \times \mathbf{v}) = \mathbf{v}\cdot(\mathbf{w} \times \mathbf{u})$$

is a number which is

- positive if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is right-handed,
- negative if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is left-handed,
- 0 if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent.

$|\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w})|$ is the volume (rumfang) of a parallelopiped where $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are three edges.

If V is a set of vectors in $\mathbb{R}^n$ satisfying

- $\mathbf{v} \in V$ and $\mathbf{w} \in V \Rightarrow \mathbf{v} + \mathbf{w} \in V$.
- $\mathbf{v} \in V$ and $c \in \mathbb{R} \Rightarrow c\mathbf{v} \in V$.

the we say that V is a subspace of $\mathbb{R}^n$.

If $\mathbf{b}_1, \ldots, \mathbf{b}_d$ are linearly independent vectors spanning V then we say that $\{\mathbf{b}_1, \ldots, \mathbf{b}_d\}$ is a basis for V.
d is then the dimensionen of V.

Subspace of dimension 0: $\{\mathbf{0}\}$
Subspace of dimension 1: line through $\mathbf{0}$.
Subspace of dimension 2: plane through $\mathbf{0}$.

Affine space of dimension 1: line (not through $\{\mathbf{0}\}$).
Affine space of dimension 2: plan (not through $\{\mathbf{0}\}$).

An affine space consists of points on the form

$$O + \mathbf{v}, \quad \mathbf{v} \in V,$$

where V is a subspace and O is a fixed point.

P_0 and P_1: two different points.
There is a unique line passing through both points. It consists of points on the form

$$tP_0 + (1-t)P_1, \quad t \in \mathbb{R}.$$

The line segment between P_0 and P_1 consists of points

$$tP_0 + (1-t)P_1, \quad \text{hvor } 0 \leq t \leq 1.$$

A set of points is said to be convex if for every pair of points P_0, P_1 in the set, the line segment between them is also contained in the set.

Let $P_0, \ldots, P_k$ be points.
The expression

$$a_0 P_0 + a_1 P_1 + \ldots + a_k P_k, \quad \text{where } a_0 + a_1 + \ldots + a_k = 1$$

is called an affine combination of $P_0, \ldots, P_k$.

The set of points that can be written as an affine combination of $P_0, \ldots P_k$ is an affine space.

$P_0, \ldots, P_k$ are said to be affinely dependent if one of the points can be written as an affine combination of the other points. Otherwise $P_0, \ldots, P_k$ are affinely independent.

W: an affine space, $P_0, \ldots, P_k \in W$.
If every point in W is an affine combination of $P_0, \ldots, P_k$ and if these points are affinely independent then we say that $P_0, \ldots, P_k$ is a simplex.

Every point P in W can then be written (in one and only one way) as

$$a_0 P_0 + a_1 P_1 + \ldots + a_k P_k, \quad \text{where } a_0 + a_1 + \ldots + a_k = 1.$$

$a_0, a_1, \ldots, a_k$ are called the barycentric coordinates for P.