# MCG - 5

A $3 \times 5$ matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 3 & 2 & 7 & -5 & 0 \\ 1 & 1 & 2 & 1 & 4 \end{bmatrix}.$$

An $m \times n$ matrix has $m$ rows, and $n$ columns.
Rows are enumerated $0, 1, \ldots, m-1$.
Columns are enumerated $0, 1, \ldots, n-1$.

The element (number) in row $i$, column $j$ is written $(A)_{ij}$ or $a_{ij}$.
In the example: $(A)_{12} = 7$.

If $A$ and $B$ are $m \times n$ matrices then $A + B$ is the $m \times n$ matrix where $(A + B)_{ij} = (A)_{ij} + (B)_{ij}$.

If $A$ is an $m \times n$ matrix and $a \in \mathbb{R}$ is a number then $aA$ is the $m \times n$ matrix where $(aA)_{ij} = a(A)_{ij}$.

$A$ an $m \times n$ matrix.
$B$ an $r \times s$ matrix.

The product $AB$ exists if $n = r$
and then the result is an $m \times s$ matrix.

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 5 & 6 & 7 & 8 \\ * & * & * & * \end{bmatrix} \begin{bmatrix} * & 1 & * & * & * & * \\ * & 2 & * & * & * & * \\ * & 3 & * & * & * & * \\ * & 4 & * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & 70 & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}$$

$70 = 5 \cdot 1 + 6 \cdot 2 + 7 \cdot 3 + 8 \cdot 4.$

Algebraic rules, a few examples:

$$A(B+C) = AB + AC$$

and

$$A(aB) = a(AB),$$

where $a$ is a number and $A, B, C$ are matrices with sizes so that the addition and multiplication is defined.

Almost all usual algebraic rules are satisfied.
Except that multiplication is not commutative:

$$AB \neq BA.$$

The transposed of an $m \times n$ matrix $A$ is an $n \times m$ matrix $A^T$ where $(A^T)_{ij} = A_{ji}$.

If

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

then

$$A^T = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}.$$

$$(A + B)^T = A^T + B^T, \quad (AB)^T = B^T A^T.$$

Identity matrix:

$$I = I_n = I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If $A$ is an $m \times n$ matrix then $AI_n = A$ and $I_mA = A$.

An $n \times 1$ matrix is a (column) vector.

A $1 \times n$ matrix is a (row) vektor. It is written as the transposed of a column vector.

Product of block matrices (if all sums and products are defined):

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}.$$

$$\begin{bmatrix} \mathbf{a}_0 & \ldots & \mathbf{a}_{n-1} \end{bmatrix} \begin{bmatrix} b_0 \\ \vdots \\ b_{n-1} \end{bmatrix} = b_0 \mathbf{a}_0 + \ldots + b_{n-1} \mathbf{a}_{n-1}.$$

$$A \begin{bmatrix} \mathbf{b}_0 & \ldots & \mathbf{b}_{n-1} \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_0 & \ldots & A\mathbf{b}_{n-1} \end{bmatrix}.$$

Let $V$ and $W$ be vector space, e.g. $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$.

A function $T : V \mapsto W$ is said to be a linear transformation if

- $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for all vectors $\mathbf{v}, \mathbf{w} \in V$, and

- $T(a\mathbf{v}) = aT(\mathbf{v})$ for all vectors $\mathbf{v} \in V$ and all numbers $a$.

**Example.** Let $\mathbf{v} = [v_x, v_y, v_z]^T \in \mathbb{R}^3$.
Then $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ defined by $T(\mathbf{x}) = \mathbf{v} \times \mathbf{x}$ is a linear transformation and $T(\mathbf{x}) = \tilde{\mathbf{v}}\mathbf{x}$ where $\tilde{\mathbf{v}}$ is the $3 \times 3$ matrix

$$\tilde{\mathbf{v}} = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}.$$

**Example.** Let $\hat{\mathbf{v}} = \in \mathbb{R}^n$, with $||\hat{\mathbf{v}}|| = 1$.
Then $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ defined by $T(\mathbf{x}) = \text{proj}_{\hat{\mathbf{v}}}\mathbf{x} = (\mathbf{x}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}$ is a linear transformation and $T(\mathbf{x}) = A\mathbf{x}$ where $A$ is the $n \times n$ matrix

$$A = \hat{\mathbf{v}}\hat{\mathbf{v}}^T = (\hat{\mathbf{v}} \otimes \hat{\mathbf{v}}).$$

In general

$$(\mathbf{v} \otimes \mathbf{w}) = \mathbf{v}\mathbf{w}^T$$

is called a tensor product.