MCG - 6

If A is an $m \times n$ matrix then the function

$$\mathcal{S} : \mathbb{R}^n \mapsto \mathbb{R}^m$$

defined by

$$\mathcal{S}(\mathbf{v}) = A\mathbf{v}$$

is a linear transformation.

If $\mathcal{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation, satisfying

$$\mathcal{T}(\mathbf{e}_0) = \mathbf{a}_0,\ \mathcal{T}(\mathbf{e}_1) = \mathbf{a}_1,\ \ldots,\ \mathcal{T}(\mathbf{e}_{n-1}) = \mathbf{a}_{n-1},$$

where

$$\mathbf{e}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{e}_{n-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

then

$$\mathcal{T}(\mathbf{v}) = A\mathbf{v},$$

where A is the matrix

$$\begin{bmatrix} \mathbf{a}_0\ \mathbf{a}_1 \ldots \mathbf{a}_{n-1} \end{bmatrix}.$$

If the linear transformation $\mathcal{S} : \mathbb{R}^p \mapsto \mathbb{R}^m$ satifies $\mathcal{S}(\mathbf{w}) = A\mathbf{w}$

and the linear transformation $\mathcal{T} : \mathbb{R}^n \mapsto \mathbb{R}^p$ satisfies $\mathcal{T}(\mathbf{v}) = B\mathbf{v}$

then

$$\mathcal{S} \circ \mathcal{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$$

is also a linear transformation and

$$(\mathcal{S} \circ \mathcal{T})(\mathbf{v}) = (AB)\mathbf{v}.$$

If $\mathcal{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation then we define the null space of $\mathcal{T}$ as

$$N(\mathcal{T}) = \{\mathbf{v} \in \mathbb{R}^n \mid \mathcal{T}(\mathbf{v}) = \mathbf{0}\}.$$

This is a subspace of $\mathbb{R}^n$.
The dimensionen of $N(\mathcal{T})$ is called *nullity*$(\mathcal{T})$.

The range of $\mathcal{T}$ is

$$R(\mathcal{T}) = \{\mathbf{w} \in \mathbb{R}^m \mid \text{there exists } \mathbf{v} \in \mathbb{R}^n \text{ so that } \mathcal{T}(\mathbf{v}) = \mathbf{w}\}.$$

This is a subspace of $\mathbb{R}^m$.
The dimension of $R(\mathcal{T})$ is called the rank of $\mathcal{T}$ and is written as *rank*$(\mathcal{T})$.

The dimensions satisfy the following equation:

$$\textit{nullity}(\mathcal{T}) + \textit{rank}(\mathcal{T}) = n.$$

A system of linear equations

$$\begin{array}{llcl} a_{00}x_0 + & a_{01}x_1 + \ldots + a_{0,n-1}x_{n-1} = & b_0 \\ a_{10}x_0 + & a_{11}x_1 + \ldots + a_{1,n-1}x_{n-1} = & b_1 \\ \vdots & & & \\ a_{m-1,0}x_0 + & a_{m-1,1}x_1 + \ldots + a_{m-1,n-1}x_{n-1} = & b_{m-1} \end{array}$$

can be denoted by its augmented coefficient matrix

$$\begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0,n-1} & b_0 \\ a_{10} & a_{11} & \cdots & a_{1,n-1} & b_1 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m-1,0} & a_{m-1,1} & \cdots & a_{m-1,n-1} & b_{m-1} \end{bmatrix}$$

Elementary row operations on matrices:

1. multiply a row by a number $k \neq 0$
2. replace row i by (row i) $+ k \cdot$ (row j), $i \neq j$
3. swap two rows.

Two $m \times n$ matrices are said to be row equivalent if one of them can be obtained from the other by using a number of elementary row operations.

Two systems of linear equations have the same set solutions if their augmented coefficient matrices are row equivalent.

A matrix is in echelon form if

1. rows with only 0's are below non-zero rows
2. the first non-zero element in a row is 1 (it is called the leading element or pivot)
3. a leading element in a row is in a column to the right of a leading element in row above it.

A matrix in echelon form is in reduced echelon form if

4. a column with a leading element (pivot) has 0 i all other rows.

Solution to a system of linear equations (when the augmented coefficient matrix is in reduced echelon form)

If the last column has a pivot then there is an equation of the form:

$$0x_0 + \ldots + 0x_{n-1} = 1,$$

and the system of equations has no solutions (it is inconsistent).

If there is a pivot in all columns except the last column then there is a unique solution to the system of equations.

If thre is no pivot in the last column and there is one more column with no pivot then there are infinitely many solutions.