MCG - 8

An **affine transformation** $\mathcal{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a function satisfying

$$\mathcal{T}(a_0 P_0 + a_1 P_1) = a_0 \mathcal{T}(P_0) + a_1 \mathcal{T}(P_1),$$

for all points P_0, P_1 and all numbers a_0, a_1 where $a_0 + a_1 = 1$.

Let $\mathcal{T}$ be an affine transformation.
Let $\mathcal{S}(\mathbf{v}) = \mathcal{T}(O + \mathbf{v}) - \mathcal{T}(O)$, where $O = (0, \ldots, 0)$.
Then $\mathcal{S}$ is a linear transformation and therefore there exists a matrix A so that $\mathcal{S}(\mathbf{v}) = A\mathbf{v}$.

The columns of A are $\mathcal{S}(\mathbf{e}_0), \ldots, \mathcal{S}(\mathbf{e}_{n-1})$.

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$$\mathcal{T}(\mathbf{v}) = A\mathbf{v} + \mathbf{y},$$

where $\mathbf{y} = \mathcal{T}(O)$.

The affine transformation is represented by the following matrix

$$\begin{bmatrix} A & \mathbf{y} \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

The inverse affine transformation $\mathcal{T}^{-1}$ is represented by the inverse matrix

$$\begin{bmatrix} A^{-1} & -A^{-1}\mathbf{y} \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

The point P in $\mathbb{R}^n$ is represented by the following vector in $\mathbb{R}^{n+1}$

$$\begin{bmatrix} P \\ 1 \end{bmatrix}.$$

A **translation** by the vector $\mathbf{t}$ maps the point P to the point $P + \mathbf{t}$.

The matrix of this affine transformation is

$$\begin{bmatrix} I_n & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

Pure **rotation**. (pure = around axis through O). The rotation is then a linear transformation.

A linear transformation $T(\mathbf{v}) = A\mathbf{v}$ is a rotation
if and only if
A is an orthogonal matrix with $\det(A) = 1$.

A composition of two rotations is a rotation.

Rotation in $\mathbb{R}^3$ around the z-axis by the angle θ has matrix

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The affine matrix is

$$\begin{bmatrix} R_z & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}.$$