MCG - 12

Multiplication of quaternions:
When computing

$$(w_2 + x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k})(w_1 + x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k})$$

we may use the following:

$$\mathbf{ij} = \mathbf{k},\ \mathbf{ji} = -\mathbf{k}, \quad \mathbf{jk} = \mathbf{i},\ \mathbf{kj} = -\mathbf{i}, \quad \mathbf{ki} = \mathbf{j},\ \mathbf{ik} = -\mathbf{j}.$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1.$$

We can also compute the product as

$$(w_2, \mathbf{v}_2)(w_1, \mathbf{v}_1) = (w_2 w_1 - \mathbf{v}_2 \cdot \mathbf{v}_1, w_1 \mathbf{v}_2 + w_2 \mathbf{v}_1 + \mathbf{v}_2 \times \mathbf{v}_1),$$

and in particular

$$(0, \mathbf{v}_2)(0, \mathbf{v}_1) = (-\mathbf{v}_2 \cdot \mathbf{v}_1, \mathbf{v}_2 \times \mathbf{v}_1).$$

All algebraic rules except the commutative law are valid. Usually:

$$q_1 q_2 \neq q_2 q_1.$$

Furthermore

$$||q_1 q_2|| = ||q_1|| \cdot ||q_2||.$$

Identity:

$$(w, \mathbf{v})(1, \mathbf{0}) = (1, \mathbf{0})(w, \mathbf{v}) = (w, \mathbf{v}).$$

Inverse: if $q = (w, \mathbf{v}) \neq (0, \mathbf{0})$ then q has inverse

$$q^{-1} = \frac{1}{||q||^2}(w, -\mathbf{v}).$$

If q is normalized ($||q|| = 1$) then

$$q^{-1} = (w, -\mathbf{v}).$$

The inverse quaternion satifies:

$$qq^{-1} = q^{-1}q = (1, \mathbf{0}).$$

Rotation by angle θ around the axis $\hat{\mathbf{r}}$ is represented by the quaternion

$$q = (\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{r}}).$$

This quaternion satisfies $||q|| = 1$.

If $\mathbf{p}$ is a vector in 3D-space then let $R_q(\mathbf{p})$ be the vector that $\mathbf{p}$ is rotated into.

We think of $\mathbf{p}$ as a quaternion, $(0, \mathbf{p})$, and then we can compute $R_q(\mathbf{p})$ as follows

$$R_q(\mathbf{p}) = q\mathbf{p}q^{-1}.$$

If $q = (w, \mathbf{v})$ then this can also be computed as

$$R_q(\mathbf{p}) = (2w^2 - 1)\mathbf{p} + 2(\mathbf{v}\cdot\mathbf{p})\mathbf{v} + 2w(\mathbf{v} \times \mathbf{p}).$$

Converting from matrix representation of rotation to quaternion representation. (page 191)

R: a rotation matrix.

Compute:
trace$(R) = R_{00} + R_{11} + R_{22}$.
$\mathbf{r} = (R_{21} - R_{12}, R_{02} - R_{20}, R_{10} - R_{01})$.

$q = (\text{trace}(R) + 1, \mathbf{r}) =$
$(R_{00} + R_{11} + R_{22} + 1, R_{21} - R_{12}, R_{02} - R_{20}, R_{10} - R_{01})$.

Then the rotationen is represented by the normalized quaternion

$$\frac{1}{||q||} q.$$