MCG - 14

Piecewise Hermite curves.

$P_0, P_1, \ldots, P_n$: points.

We want to find Hermite curves
$Q_0(u), Q_1(u), \ldots, Q_{n-1}(u)$,
so that each $Q_i(u)$ is a curve moving from P_i to P_{i+1} when u increases from 0 to 1:
$Q_i(0) = P_i$ and $Q_i(1) = P_{i+1}$ for all $i = 0, 1, \ldots, n-1$

In order to compute a unique Hermite $Q_0(u)$ we need to choose vectors $\mathbf{P}'_0$ and $\mathbf{P}'_1$ and require that $\mathbf{Q}'(0) = \mathbf{P}'_0$ and $\mathbf{Q}'(1) = \mathbf{P}'_1$ and similar for the other curves.

We may also use an automatic way to generate these conditions.

Automatic generation of Hermite curves.

A way to generate a piecewise Hermite curve through the points $P_0, P_1, \ldots, P_n$. The following conditions must be satisfied:

- $Q_i(1) = Q_{i+1}(0) = P_i$ for all $i = 0, 1, \ldots, n-2$ (previous slide)
- $\mathbf{Q}'_i(1) = \mathbf{Q}'_{i+1}(0)$ for all $i = 0, 1, \ldots, n-2$
  Q_{i+1} starts with the same velocity as Q_i has in the end.
- $\mathbf{Q}''_i(1) = \mathbf{Q}''_{i+1}(0)$ for alle $i = 0, 1, \ldots, n-2$
  Q_{i+1} starts with the same acceleration as Q_i has in the end.
- $\mathbf{Q}''_0(0) = \mathbf{0}$ og $\mathbf{Q}''_{n-1}(1) = \mathbf{0}$ (natural end conditions).
  No acceleration in the beginning and at the end.

In order to determine $\mathbf{P}_0' = \mathbf{Q}_0'(0), \mathbf{P}_1' = \mathbf{Q}_1'(0) = \mathbf{Q}_0'(1), \ldots, \mathbf{P}_{n-1}' = \mathbf{Q}_{n-1}'(0) = \mathbf{Q}_{n-2}'(1), \mathbf{P}_n' = \mathbf{Q}_{n-1}'(1)$ we derive the following system of equations from the equations on the previous slide (the matrix has size $(n+1) \times (n+1)$):

$$\begin{bmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & \cdots & 1 & 4 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 4 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{P}_0' \\ \mathbf{P}_1' \\ \mathbf{P}_2' \\ \vdots \\ \mathbf{P}_{n-2}' \\ \mathbf{P}_{n-1}' \\ \mathbf{P}_n' \end{bmatrix} = \begin{bmatrix} 3(P_1 - P_0) \\ 3(P_2 - P_0) \\ 3(P_3 - P_1) \\ \vdots \\ 3(P_{n-1} - P_{n-3}) \\ 3(P_n - P_{n-2}) \\ 3(P_n - P_{n-1}) \end{bmatrix}.$$

When $\mathbf{P}'_0, \mathbf{P}'_1, \ldots, \mathbf{P}'_n$ have been determined from the above equtions we can compute each Q_i as follows:

$$Q_i(u) = UMG,$$

where

$$U = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} P_i \\ P_{i+1} \\ \mathbf{P}'_i \\ \mathbf{P}'_{i+1} \end{bmatrix}.$$

We have the following general formula:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i, \quad \text{hvor} \quad \binom{n}{i} = \frac{n!}{i!(n-i)!},$$

and $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.

In particular $(x+y)^2 = x^2 + 2xy + y^2$ and

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

If we let $x = 1 - u$ and $y = u$ we get

$$1 = (1-u)^3 + 3(1-u)^2u + 3(1-u)u^2 + u^3.$$

Bézier kurver:

$P_0, P_1, \ldots, P_n$ are points, called control points.

The Bézier curve is then

$$Q(u) = \sum_{i=0}^{n} \binom{n}{i} (1-u)^{n-i} u^i P_i.$$

It satisfies $Q(0) = P_0$ and $Q(1) = P_n$.

The most interesting case is $n = 3$:

$$Q(u) = (1-u)^3 P_0 + 3u(1-u)^2 P_1 + 3u^2(1-u) P_2 + u^3 P_3.$$

In the case $n = 3$ the Bézier curve can also be written as

$$Q(u) = J_{3,0}(u)P_0 + J_{3,1}(u)P_1 + J_{3,2}(u)P_2 + J_{3,3}(u)P_3,$$

where

$J_{3,0}(u) = (1-u)^3 = 1 - 3u + 3u^2 - u^3$
$J_{3,1}(u) = 3u(1-u)^2 = 3u - 6u^2 + 3u^3$
$J_{3,2}(u) = 3u^2(1-u) = 3u^2 - 3u^3$
$J_{3,3}(u) = u^3$

As $J_{3,0}(u) + J_{3,1}(u) + J_{3,2}(u) + J_{3,3}(u) = 1$ $Q(u)$ is an affine combination of P_0, P_1, P_2, P_3.

Furthermore $J_{3,0}(u) \geq 0$, $J_{3,1}(u) \geq 0$, $J_{3,2}(u) \geq 0$ and $J_{3,3}(u) \geq 0$. Thus $Q(u)$ is a convex combination of P_0, P_1, P_2, P_3 and the curve is contained in the convex hull of P_0, P_1, P_2, P_3.

The Bézier curve with $n = 3$ satifies $\mathbf{Q}'(0) = 3(P_1 - P_0)$ and $\mathbf{Q}'(1) = 3(P_3 - P_2)$ and the curve is the same as a Hermite curve

$$Q(u) = UMG,$$

where

$$U = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} P_0 \\ P_3 \\ 3(P_1 - P_0) \\ 3(P_3 - P_2) \end{bmatrix}.$$