Reparametrizations of Continuous Paths

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Paths and Reparametrizations in Differential Geometry

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path: $p : I \rightarrow \mathbf{R}^n$, continuous, differentiable on $(0,1)$.
regular path: $p'(t) \neq \mathbf{0}$, $t \in (0,1)$.

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reparametrization: $\varphi : (I; 0, 1) \to (I, 0, 1)$, differentiable in $(0, 1)$, and $\varphi'(t) \neq 0$, i.e., strictly increasing.
Consequence: For every path p and every reparametrization φ, p and $p \circ \varphi$ have the same trace.
In differential geometry, one investigates invariants of the traces = reparametrization equivalence classes of paths—e.g., curvature, torsion.

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regular?, reparametrization?

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- *Reparametrization equivalence is an equivalence relation.*

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Theorem

- *Reparametrization equivalence is an equivalence relation.*
- *Every path is reparametrization equivalent to a regular path.*

Tools: Stop intervals, stop values, stop map etc.

Definition

- Given a path p in X. An interval $[a, b] \subset I$ is a *p-stop interval* if it is a maximal interval on which p is constant.
- Let $\mathfrak{P}_{[\,]}(I) = \{[a, b] \mid 0 \le a < b \le 1\}$. Then $\Delta(p) \subset \mathfrak{P}_{[\,]}(I)$ is the (ordered) subset consisting of all p-stop intervals.

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- An element $c \in X$ is a p-stop value if there is a p-stop interval $J \in \Delta_p$ with $p(J) = \{c\}$. We let $C_p \subseteq X$ denote the set of all p-stop values.

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Proposition

The sets Δ_p and C_p are at most countable.

Reparametrizations in their own right

$Inc_+(I) := \{\varphi : (I;0,1) \to (I;0,1) \mid \varphi$ increasing $\} \supset$

$Rep_+(I) := \{\varphi \in Inc_+(I) \mid \varphi$ continuous $\} \supset$ a monoid

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Fact

An element $\varphi \in Inc_+(I)$ is contained in

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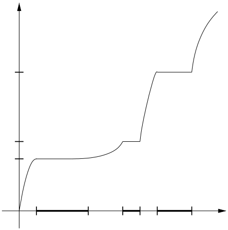

Figure: Stop intervals and stop values of a reparametrization

All countable sets are stop value sets!

Proposition

For every (at most) countable set $C \subset I$*, there is a reparametrization* $\varphi \in Rep_+(I)$ *with* $C_\varphi = C$*.*

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Proposition

For every (at most) countable set $C \subset I$, there is a reparametrization $\varphi \in Rep_+(I)$ with $C_\varphi = C$.

Proof.

1. Construct a sequence of piecewise linear maps $\varphi_n \in Rep_+(I)$

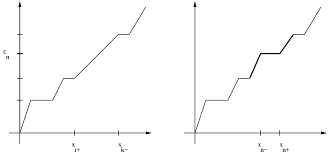

Figure: Inserting the stop value c_n

inserting one stop value c_n at a time.

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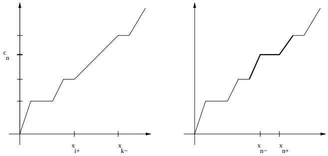

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2. Make sure that $\| \varphi_{n+1} - \varphi_n \| < \frac{1}{2^n}$ to ensure uniform convergence.

Given

- an at most countable subset $\Delta \subset \mathfrak{P}_{[]}(I)$ of disjoint closed intervals
- an at most countable subset $C \subset I$ and
- an order-preserving bijection $F : \Delta \to C$

Classification of Reparametrizations - Questions

Focusing on the combinatorial data

Given

- an at most countable subset $\Delta \subset \mathfrak{P}_{[]}(I)$ of disjoint closed intervals
- an at most countable subset $C \subset I$ and
- an order-preserving bijection $F : \Delta \to C$

1. Is there a reparametrization $\varphi \in Rep_+(I)$ suc that $\Delta_\varphi = \Delta, C_\varphi = C, F_\varphi = F$?
2. How many?

Classification of Reparametrizations - Answers

1. Yes iff the following conditions are met for $J_n, K_m \in \Delta$:
   For $\max J_n \uparrow x \in I, \min K_n \downarrow y \in I$:
   1.1 $x = y \Rightarrow \lim F(J_n) = \lim F(K_n)$,
   1.2 $x < y \Rightarrow \lim F(J_n) < \lim F(K_n)$,
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2. Let $D = \bigcup_{J \in \Delta}$ (the stop set), $O = I \setminus \bar{D} = \bigcup_{J \in \Gamma}$ (the move set, Js: disjoint maximal open intervals):
   The set of reparametrizations with stop map F is in one-to-one correspondence with $\prod_{J \in \Gamma} Homeo_+(I)$.

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The first answer above opens up for a combinatorial study of reparametrizations.

1. Given $\varphi, \psi \in Rep_+(I)$ with associated stop maps $F_\varphi : \Delta_\varphi \to C_\varphi, F_\psi : \Delta_\psi \to C_\psi$.
   Composition: Give a description of $\Delta_{\phi\circ\psi}, C_{\phi\circ\psi}, F_{\phi\circ\psi}$.

Algebra: Compositions and Factorizations – Questions
through stop maps

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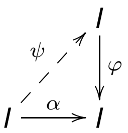
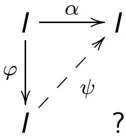
2. Let $\alpha, \varphi \in Rep_+(I)$.
   Under which conditions are there factorizations



resp.



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Compositions and Factorizations – Selected Answers

1. $C_{\varphi\circ\psi} = \phi(C_\psi) \cup C_\varphi$.

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   2.1 if and only if $C_\varphi \subseteq C_\alpha$.
   In that case C_ψ can be chosen arbitrarily in the range
   $\varphi^{-1}(C_\alpha \setminus C_\varphi) \subseteq C_\psi \subseteq \varphi^{-1}(C_\alpha \setminus C_\varphi) \cup D_\varphi$.

   2.2 if and only if there exists a map $i_{\varphi\alpha} : \Delta_\varphi \to \Delta_\alpha$ such that $J \subseteq i_{\varphi\alpha}(J)$ for every $J \in \Delta_\varphi$. (Δ_φ is a *refinement* of Δ_α).
   If so, ψ is uniquely determined.

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Lifts are constructed using the stop maps discussed previously.

The algebra of reparametrizations up to homeomorphisms

Consider the group action
$Rep_+(I) \times Homeo_+(I) \to Rep_+(I),\ (\varphi, \psi) \mapsto \varphi \circ \psi$.
Along an orbit, the stop values are preserved.

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More precisely: let $\mathfrak{P}_c(I)$ denote the set of *countable* subsets of I.

Proposition

$C : Rep_+(I)/_{Homeo_+(I)} \to \mathfrak{P}_c(I),\ C([\alpha]) = C_\alpha$ is a bijection.

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The map C becomes even an isomorphism of distributive lattices with natural operations corresponding to $\cup, \cap$, e.g.:

Proposition

For every $\varphi_1, \varphi_2 \in Rep_+(I)$, there exist $\psi_1, \psi_2 \in Rep_+(I)$

completing the diagram

$$\begin{array}{ccc} I & \overset{\psi_1}{\dashrightarrow} & I \\ {\scriptstyle \psi_2}\downarrow & & \downarrow{\scriptstyle \varphi_1} \\ I & \underset{\varphi_2}{\longrightarrow} & I \end{array}$$

with $C_{\varphi_1 \circ \psi_1} = C_{\varphi_2 \circ \psi_2} = C_{\varphi_1} \cup C_{\varphi_2}$.

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Proof.

Transitivity: Assume $p \circ \varphi = q \circ \psi$ and $q \circ \varphi' = r \circ \psi'$ for paths $p, q, r \in P(X)$ and reparametrizations $\varphi, \varphi', \psi, \psi' \in Rep_+(I)$.
There are reparametrizations $\eta, \eta' \in Rep_+(I)$ such that $\psi \circ \eta = \varphi' \circ \eta'$; hence $p \circ \varphi \circ \eta = r \circ \psi' \circ \eta'$. □

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(using results on reparametrizations!)

1. Define $m : \Delta_p \to I$ the "midpoint" map on the stop intervals; $C = m(\Delta_p) \subset I$.

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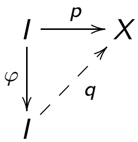
3. There is a factorization 

through a regular path q.

   Check continuity!

Traces = Regular Traces

$P(X)(x,y)$ space of paths p in X with $p(0)=x$ and $p(y)=1$
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Corollary

$R(X)(x,y)$ and $T(X)(x,y)$ are homotopy equivalent.

Traces = Regular Traces on saturated d-spaces

d-space (Grandis):
a topological space with a set $\vec{P}(X) \subset P(X)$ of directed paths, closed under (increasing) reparametrizations and under concatenation.

saturated *d*-space:
If $p \in P(X), \varphi \in Rep_+(I)$ and $p \circ \varphi \in \vec{P}(X)$, then $p \in \vec{P}(X)$.

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Corollary

Let X denote a saturated d-space and let $x, y \in X$. The map $\vec{i}: \vec{T}_R(X)(x,y) \to \vec{T}(X)(x,y)$ induced by inclusion $\vec{R}(X)(x,y) \hookrightarrow \vec{P}(X)(x,y)$ is a homeomorphism.

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d-spaces are used to model concurrency geometrically—directed paths model concurrent executions. The Corollary opens up for an algebraic topological and categorical investigation of the spaces of traces (concatenation is associative!) via functors from algebraic topology (homotopy, homology etc.)