Concurrency and directed algebraic topology

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M18: Minisymposium
Computational and Combinatorial Algebraic Topology
Jahrestagung DMV-GDM
Berlin, 29.3.2007

Outline

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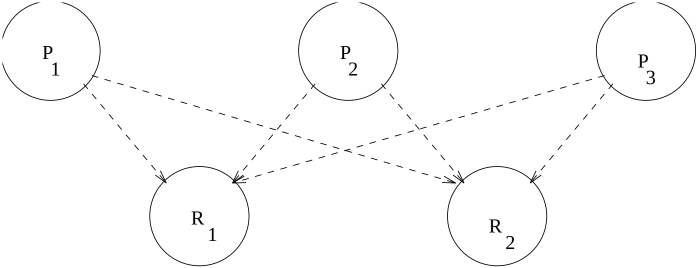
1. Motivations, mainly from Concurrency Theory
2. Directed topology: algebraic topology with a twist
3. A categorical framework (with examples)
4. “Compression” of ditopological categories: generalized congruences via homotopy flows

Main Collaborators:

- Lisbeth Fajstrup (Aalborg), Éric Goubault, Emmanuel Haucourt (CEA, France)

Mutual exclusion occurs, when n processes P_i compete for m resources R_j.



Only k processes can be served at any given time.

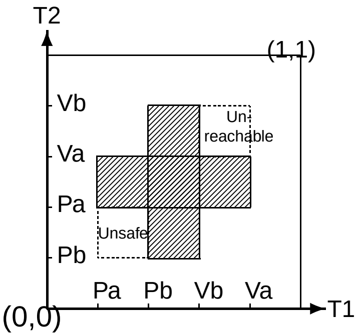
Semaphores!

Semantics: A processor has to lock a resource and relinquish the lock later on!

Description/abstraction $P_i : \ldots PR_j \ldots VR_j \ldots$ (Dijkstra)

Schedules in "progress graphs"

The Swiss flag example



PV-diagram from

$P_1 : P_aP_bV_bV_a$

$P_2 : P_bP_aV_aV_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded).

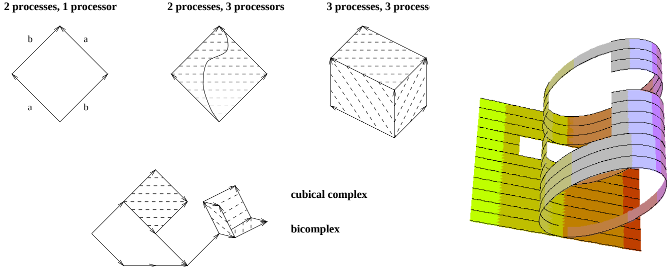
Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions.

Deadlocks, unsafe and unreachable regions may occur.

Higher dimensional automata

seen as (geometric realizations of) cubical sets

Vaughan Pratt, Rob van Glabbeek, Eric Goubault...



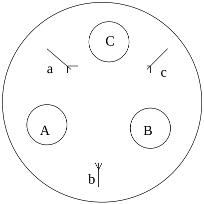
Squares/cubes/hypercubes are filled in iff actions on boundary are independent.

Higher dimensional automata are cubical sets:

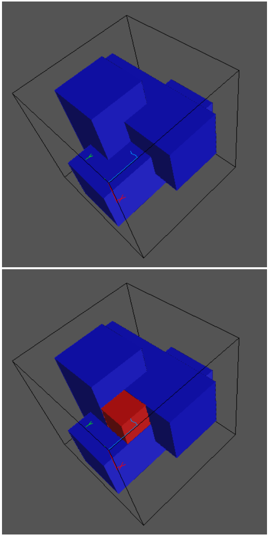
- like simplicial sets, but modelled on (hyper)cubes instead of simplices; glueing by face maps (and degeneracies)
- additionally: preferred directions – not all paths allowable.

Higher dimensional automata

Dining philosophers



A=Pa.Pb.Va.Vb
B=Pb.Pc.Vb.Vc
C=Pc.Pa.Vc.Va



Higher dimensional complex with a forbidden region consisting of isothetic hypercubes and an unsafe region.

Discrete versus continuous models

How to handle the state-space explosion problem?

Discrete models for concurrency (transition graph models) suffer a severe problem if the number of processors and/or the length of programs grows: The number of states (and the number of possible schedules) grows exponentially: this is known as the state space explosion problem.

You need clever ways to find out which of the schedules yield equivalent results – e.g., to check for correctness – for general reasons.

Alternative: Infinite continuous models allowing for well-known equivalence relations on paths (homotopy = 1-parameter deformations) – but with an important twist!

Analogy: Continuous physics as an approximation to (discrete) quantum physics.

A framework for directed topology
d-spaces, M. Grandis (03)

X a topological space. $\vec{P}(X) \subseteq X^I = \{p : I = [0,1] \to X \text{ cont.}\}$
a set of d-paths ("directed" paths $\leftrightarrow$ executions) satisfying

- $\{$ constant paths $\} \subseteq \vec{P}(X)$
- $\varphi \in \vec{P}(X)(x,y), \psi \in \vec{P}(X)(y,z) \Rightarrow \varphi * \psi \in \vec{P}(X)(x,z)$
- $\varphi \in \vec{P}(X), \alpha \in I^I$ a nondecreasing reparametrization $\Rightarrow \varphi \circ \alpha \in \vec{P}(X)$

The pair $(X, \vec{P}(X))$ is called a d-space.
Observe: $\vec{P}(X)$ is in general not closed under reversal:

$$\alpha(t) = 1 - t,\ \varphi \in \vec{P}(X) \not\Rightarrow \varphi \circ \alpha \in \vec{P}(X)!$$

Examples:

- An HDA with directed execution paths.
- A space-time(relativity) with time-like or causal curves.

D-maps, Dihomotopy, d-homotopy

A d-map $f : X \to Y$ is a continuous map satisfying

- $f(\vec{P}(X)) \subseteq \vec{P}(Y)$

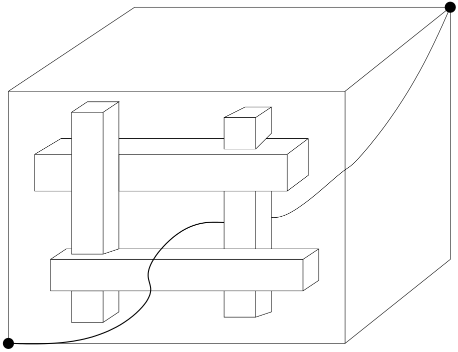
special case: $\vec{P}(I) = \{\sigma \in I^I | \sigma$ nondecreasing reparametrization$\}$, $\vec{I} = (I, \vec{P}(I))$.
Then $\vec{P}(X) =$ set of d-maps from $\vec{I}$ to X.

- Dihomotopy $H : X \times I \to Y$, every H_t a d-map
- elementary d-homotopy = d-map $H : X \times \vec{I} \to Y$ – $H_0 = f \xrightarrow{H} g = H_1$
- d-homotopy: symmetric and transitive closure (”zig-zag”)

L. Fajstrup, 05: In cubical models (for concurrency, e.g., HDAs), the two notions agree for d-paths ($X = \vec{I}$). In general, they do not.

Dihomotopy is finer than homotopy with fixed endpoints

Example: Two wedges in the forbidden region

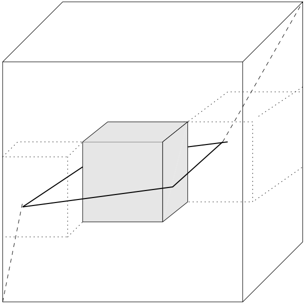


All dipaths from minimum to maximum are homotopic.
A dipath through the “hole” is not dihomotopic to a dipath on the boundary.

The twist has a price

Neither homogeneity nor cancellation nor group structure

In ordinary topology, it suffices to study loops in a space X with a given start=end point x (one per path component). Moreover: "Loops up to homotopy" $\rightsquigarrow$ fundamental group $\pi_1(X,x)$ – concatenation, inversion!



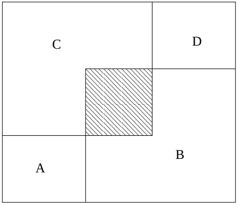
"Birth and death" of dihomotopy classes

Directed topology: Loops do not tell much; concatenation ok, cancellation not! Replace group structure by category structures!

A first remedy: the fundamental category

$\vec{\pi}_1(X)$ of a d-space X [Grandis:03, FGHR:04]:

- Objects: points in X
- Morphisms: d- or dihomotopy classes of d-paths in X
- Composition: from concatenation of d-paths



Property: van Kampen theorem (M. Grandis)
Drawbacks: Infinitely many objects. Calculations?
Question: How much does $\vec{\pi}_1(X)(x,y)$ depend on (x,y)?
Remedy: Localization, component category. [FGHR:04, GH:06]
Problem: “Compression” works only for loopfree categories

Technique: Traces – and trace categories

Getting rid of increasing reparametrizations

X a (saturated) d-space.
$\varphi, \psi \in \vec{P}(X)(x,y)$ are called reparametrization equivalent if there are $\alpha, \beta \in \vec{P}(I)$ such that $\varphi \circ \alpha = \psi \circ \beta$.
(Fahrenberg-R., 07): Reparametrization equivalence is an equivalence relation (transitivity).
$\vec{T}(X)(x,y) = \vec{P}(X)(x,y)/_{\simeq}$ makes $\vec{T}(X)$ into the (topologically enriched) trace category – composition associative.
A d-map $f : X \to Y$ induces a functor $\vec{T}(f) : \vec{T}(X) \to \vec{T}(Y)$.
Variant: $\vec{R}(X)(x,y)$ consists of regular d-paths (not constant on any non-trivial interval $J \subset I$). The contractible group $Homeo_+(I)$ of increasing homeomorphisms acts on these – freely if $x \neq y$.

Theorem (FR:07)

$\vec{R}(X)(x,y)/_{\simeq} \to \vec{P}(X)(x,y)/_{\simeq}$ is a homeomorphism.

Preorder categories
Getting organised with indexing categories

A d-space structure on X induces the preorder $\preceq$:

$$x \preceq y \Leftrightarrow \vec{T}(X)(x,y) \neq \emptyset$$

and an indexing preorder category $\vec{D}(X)$ with

- Objects: (end point) pairs $(x,y), x \preceq y$
- Morphisms:
  $\vec{D}(X)((x,y),(x',y')) := \vec{T}(X)(x',x) \times \vec{T}(X)(y,y')$:

$$\begin{tikzcd} x' \arrow[r, shift left] \arrow[r, shift right] & x \arrow[r, "\preceq"] & y \arrow[r, shift left] \arrow[r, shift right] & y' \end{tikzcd}$$

- Composition: by pairwise contra-, resp. covariant concatenation.

A d-map $f : X \to Y$ induces a functor $\vec{D}(f) : \vec{D}(X) \to \vec{D}(Y)$.

The trace space functor
Preorder categories organise the trace spaces

The preorder category organises X via the
trace space functor $\vec{T}^X : \vec{D}(X) \to Top$

- $\vec{T}^X(x,y) := \vec{T}(X)(x,y)$
- $\vec{T}^X(\sigma_x,\sigma_y):$

$$\begin{aligned} \vec{T}(X)(x,y) &\longrightarrow \vec{T}(X)(x',y') \\ [\sigma] &\longmapsto [\sigma_x * \sigma * \sigma_y] \end{aligned}$$

Homotopical variant $\vec{D}_\pi(X)$ with morphisms
$\vec{D}_\pi(X)((x,y),(x',y')) := \vec{\pi}_1(X)(x',x) \times \vec{\pi}_1(X)(y,y')$
and trace space functor $\vec{T}_\pi^X : \vec{D}_\pi(X) \to Ho-Top$ (with homotopy classes as morphisms).

Sensitivity with respect to variations of end points
A persistence point of view

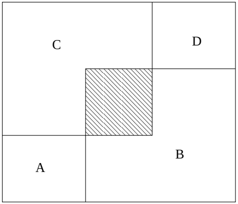
- How much does (the homotopy type of) $\vec{T}^X(x,y)$ depend on (small) changes of x,y?
- Which concatenation maps $\vec{T}^X(\sigma_x,\sigma_y):\vec{T}^X(x,y)\to\vec{T}^X(x',y'),\ [\sigma]\mapsto[\sigma_x*\sigma*\sigma_y]$ are homotopy equivalences, induce isos on homotopy, homology groups etc.?
- The persistence point of view: Homology classes etc. are born (at certain branchings/mergings) and may die (analogous to the framework of G. Carlsson etal.)
- Are there components with (homotopically/homologically) stable dipath spaces (between them)? Are there borders ("walls") at which changes occur?
- $\rightsquigarrow$ need a lot of bookkeeping!

Dihomology $\vec{H}_*$

- For every d-space X, there are homology functors

$$\vec{H}_{*+1}(X) = H_* \circ \vec{T}_\pi^X : \vec{D}_\pi(X) \to Ab,\ (x,y) \mapsto H_*(\vec{T}(X)(x,y))$$

  capturing homology of all relevant d-path spaces in X and the effects of the concatenation structure maps.
- A d-map $f : X \to Y$ induces a natural transformation $\vec{H}_{*+1}(f)$ from $\vec{H}_{*+1}(X)$ to $\vec{H}_{*+1}(Y)$.
- Properties? Calculations? Not much known in general. A master's student has studied this topic for X a cubical complex (its geometric realization) by constructing a cubical model for d-path spaces.
- Higher dihomotopy functors $\vec{\pi}_*$: in the same vain, a bit more complicated to define, since they have to reflect choices of base paths.



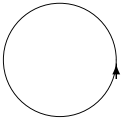
$$
\begin{tikzcd}
AA \arrow[drr] \arrow[ddr] & & & & BB \arrow[dll] \arrow[ddl] \\
& & AB \arrow[d] & \# & \\
& AC \arrow[r] & AD & BD \arrow[l] & \\
& \# & CD \arrow[u] & & \\
CC \arrow[uur] \arrow[urr] & & & & DD \arrow[ull] \arrow[uul]
\end{tikzcd}
$$

Figur: Standard example and component category

Components A, B, C, D – or rather $AA, AB, AC, AD, BB, BD, CC, CD, DD \subseteq X \times X$.
#: diagram commutes.

Examples of component categories
Oriented circle

$$X \quad = \quad \vec{S}^1$$



oriented circle

$$\mathcal{C} : \Delta \underset{b}{\overset{a}{\rightleftarrows}} \bar{\Delta}$$

Δ the diagonal, $\bar{\Delta}$ its complement.
$\mathcal{C}$ is the free category generated by a, b.

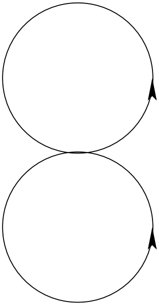
- Remark that the components are no longer products!
- It is essential in order to get a discrete component category to use an indexing category taking care of pairs (source, target).

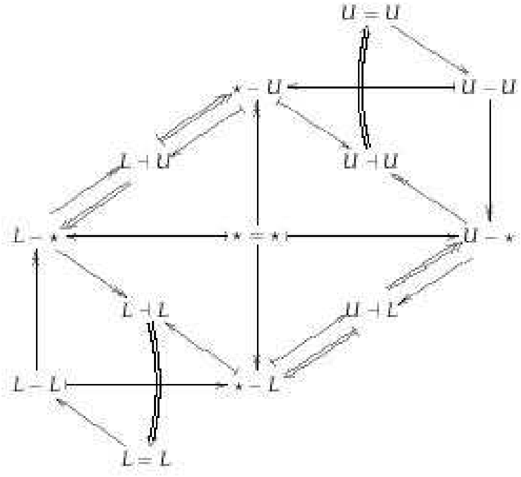
Component categories
via generalized congruences and homotopy flows

- How to identify morphisms in a category between different objects in an organised manner?
  Generalized congruence (Bednarczyk, Borzyszkowski, Pawlowski, TAC 1999) $\rightsquigarrow$ quotient category.
- Homotopy flows identify both elements and d-paths: Like flows in differential geometry. Instead of diffeotopies: Self-homotopies inducing homotopy equivalences on spaces of d-paths with given end points ("automorphic").
- Homotopy flows give rise to significant generalized congruences. Corresponding component category $\vec{D}_\pi(X)/\simeq$ identifies pairs of points on the same "homotopy flow line" and (chains of) morphisms.

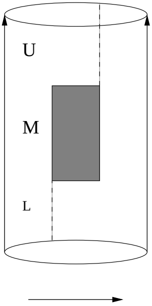
The component category of a wedge of two oriented circles

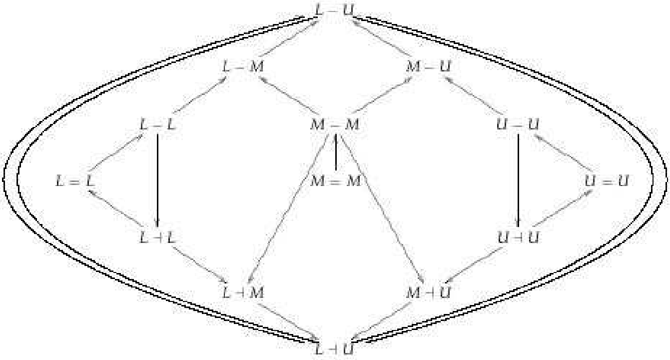
$X = \vec{S}^1 \vee \vec{S}^1$





The component category of an oriented cylinder with a deleted rectangle





Concluding remarks

- Component categories contain the essential information given by (algebraic topological invariants of) path spaces
- Compression via component categories as an antidote to the state space explosion problem
- Some of the ideas (for the fundamental category) are implemented and have been tested for huge industrial software from EDF (Éric Goubault & Co., CEA)
- Dihomotopy equivalence: Definition uses automorphic homotopy flows to ensure homotopy equivalences

$$\vec{T}(f)(x,y) : \vec{T}(X)(x,y) \to \vec{T}(Y)(fx,fy) \text{ for all } x \preceq y.$$

- Much more theoretical and practical work remains to be done!