Directed algebraic topology and applications

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Discrete Structures in Algebra, Geometry, Topology and Computer Science

6ECM July 3, 2012

Homotopy: 1-parameter deformation

- Two continuous functions $f, g : X \to Y$ from a topological space X to another, Y are called homotopic if one can be "continuously deformed" into the other.
- Such a deformation is called a homotopy $H : X \times I \to Y$ between the two functions.
- Two spaces X, Y are called homotopy equivalent if there are continous maps $f : X \to Y$ and $g : Y \to X$ that are homotopy inverse to each other, i.e., such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$.

- Algebraic topology is the branch of mathematics which uses tools from abstract algebra to study topological spaces. The basic goal is to find algebraic invariants that classify topological spaces up to homeomorphism, though usually most classify (at best) up to homotopy equivalence.
- An outstanding use of homotopy is the definition of homotopy groups $\pi_n(X, *),\ n > 0$ – important invariants in algebraic topology.

Examples

- Spheres of different dimensions are not homotopy equivalent to each other.
- Euclidean spaces of different dimensions are not homeomorphic to each other.

Path spaces, loop spaces and homotopy groups

Definition

Path space $P(X)(x_0, x_1)$: the space of all continuous paths $p : I \to X$ starting at x_0 and ending at x_1 (CO-topology).

Loop space $\Omega(X)(x_0)$: the space of all all continuous loops $\omega : S^1 \to X$ starting and ending at x_0.

Concatenation: $P(X)(x_0, x_1) \times P(X)(x_1, x_2) \to P(X)(x_0, x_2)$; $\Omega(X)(x_0) \times \Omega(X)(x_0) \to \Omega(X)(x_0)$.

Free path space, loop space $P(X), \Omega(X)$: consists of all paths/loops; no restriction on end points.

Easy facts

X a reasonable path-connected space, then

- $P(X)(x_0, x_1) \simeq P(X)(x'_0, x'_1) \simeq \Omega(X)(x_0)$.
- $\pi_n(X; x_0) \cong \pi_{n-1}(\Omega X; x_0), n > 0$.

X a topological space.

Definition

- $\vec{P}(X) \subset P(X)$ a subspace of d-paths
  - containing constant paths
  - closed under concatenation and
  - subpaths and increasing reparametrizations $I \to I \xrightarrow{p} X$.
- $(X, \vec{P}(X))$ is called a d-space.
- A continous map $F : X \to Y$ between d-spaces is a d-map if $F(\vec{P}X) \subseteq \vec{P}(Y)$.
- A homotopy $H : X \times I \to Y$ is a d-homotopy if each $H_t, \quad 0 \leq t \leq 1, \quad$ is a d-map.

Symmetry breaking

The reverse of a d-path need not be a d-path.
$\rightsquigarrow$ less structure on algebraic invariants.

Examples of d-spaces

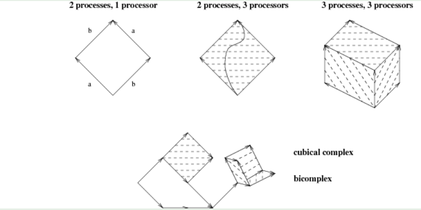
Simple examples

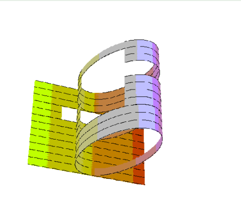
- $X = \mathbf{R}^n$, $\vec{P}(\mathbf{R}^n)$ all paths with non-decreasing components.
- $Y = I^n$, $\vec{P}(I^n)$ as above.
- $X = S^1$, $\vec{P}(S^1)$ all paths that rotate counter-clockwise.

Higher Dimensional Automata = cubical complexes

Like simplicial complexes, glued from hypercubes I^n instead of simplices; d-paths non-decreasing on every hypercube.

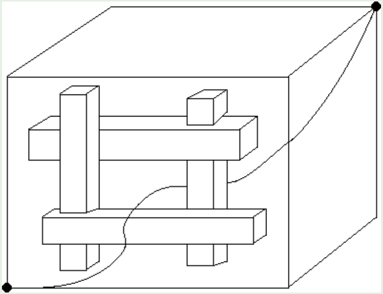
Example





Homotopic d-paths need not be d-homotopic!

A 3D-cube with two wedges deleted ($\simeq S^2 \vee S^2$)



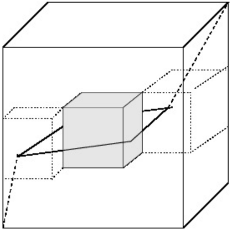
All dipaths from bottom to top are homotopic.
A dipath through the "hole" is not d-homotopic to a dipath on the boundary.

The twist has a price

Neither homogeneity nor cancellation nor group structure

Ordinary topology

- Path space = loop space (within each path component).
- A loop space is an *H*-space with concatenation, inversion, cancellation.



"Birth and death" of d-homotopy classes

Directed topology

Loops do not tell much; concatenation ok, cancellation not! Replace group structure by category structures!

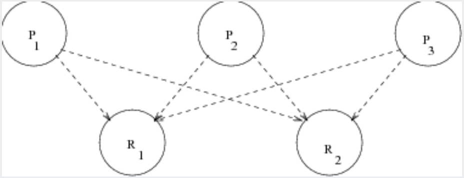
Why bother: Concurrency
Definition from Wikipedia

Concurrency

- In computer science, concurrency is a property of systems in which several computations are executing simultaneously, and potentially interacting with each other.
- The computations may be executing on multiple cores in the same chip, preemptively time-shared threads on the same processor, or executed on physically separated processors.
- A number of mathematical models have been developed for general concurrent computation including Petri nets, process calculi, the Parallel Random Access Machine model, the Actor model and the Reo Coordination Language.
- Specific applications to static program analysis – design of automated tools to test correctness etc. of a concurrent program regardless of specific timed execution.

Mutual exclusion

occurs, when n processes P_i compete for m resources R_j.



Only k processes can be served at any given time.

Semaphores

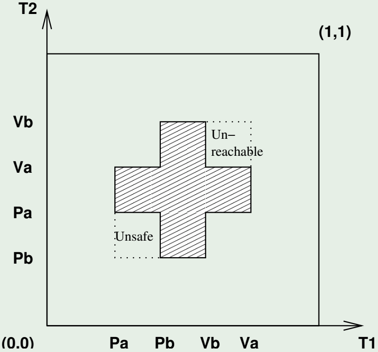
Semantics: A processor has to lock a resource and to relinquish the lock later on!

Description/abstraction: $P_i : \dots PR_j \dots VR_j \dots$ (E.W. Dijkstra)

P: probeer; V: verhoog

A geometric model: Schedules in "progress graphs"

Semaphores: The Swiss flag example



PV-diagram from
$P_1 : P_a P_b V_b V_a$
$P_2 : P_b P_a V_a V_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded). Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions. Deadlocks, unsafe and unreachable regions may occur.

The state space

A linear PV-program is modeled as the complement of a forbidden region F consisting of a number of holes in an n-cube:
Hole = isothetic hyperrectangle
$R^i =]a^i_1, b^i_1[\times \cdots \times]a^i_n, b^i_n[\subset I^n, 1 \leq i \leq l$:
with minimal vertex $\mathbf{a}^i$ and maximal vertex $\mathbf{b}^i$.
State space $X = \vec{I}^n \setminus F, \; F = \bigcup_{i=1}^{l} R^i$
X inherits a partial order from $\vec{I}^n$. d-paths are order preserving.

More general programs:

Cubical complexes: The local partial order giving rise to the d-space structure models the directed time flow.

Aims

A list of aims

- Structure and determine the d-path spaces $\vec{P}(X)(x_0, x_1)$ for reasonable d-spaces X – as ordinary topological spaces.
- Describe the path category $\vec{P}(X)$
  - Objects: points
  - Morphisms: (Homotopy types of) d-path spaces with given end points

  and reason about sensitivity with respect to end points.
- Investigate directed coverings as geometric counterparts for simulations of concurrent systems.

Nerve lemma

Given an open covering $\mathcal{U}$ of a space X such that every non-empty intersection of sets in $\mathcal{U}$ is contractible, then $X \simeq \mathcal{N}(\mathcal{U})$ – the nerve of the covering:
A simplicial complex with one n-simplex for every non-empty intersection of $n+1$ sets in $\mathcal{U}$.

General idea: HDA without d-loops

- Find decomposition of state space into subspaces so that d-path spaces in each piece – and intersections of such – are either contractible or empty.
- Describe the poset category corresponding to non-empty intersections using binary matrices.

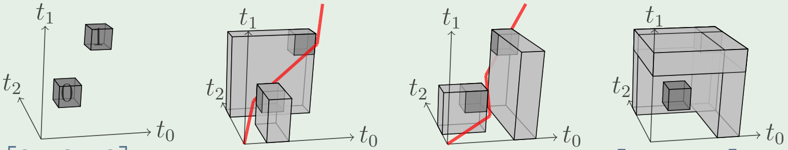
HDA with d-loops

- L_1-length yields a homomorphism $l : \pi_1(X) \to \mathbf{Z}$.
- The associated length covering $\tilde{X}$ has only trivial d-loops.
- $\vec{P}(X)(x_0, x_1) \simeq \bigsqcup_n \vec{P}(X)(\tilde{x}_0, \tilde{x}_1^n)$

Example: A 3D-cube with two subcubes deleted

Category of binary matrices describes contractible or empty subspaces

$P_a.V_a.P_b.V_b \mid P_a.V_a.P_b.V_b \mid P_a.V_a.P_b.V_b$



$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

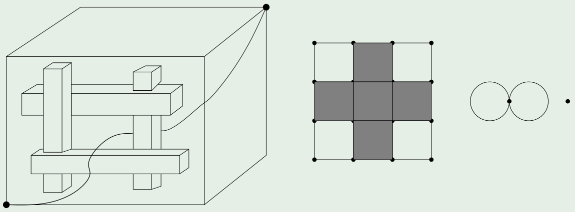
state space | alive | alive | dead

Poset category and realization

$\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = \{M \in M_{2,3}(\mathbf{Z}/2) \mid$ no row = $[0,0,0]$ or $[1,1,1]\}$.
Associated (prod-)simplicial complex: $S^1 \times S^1$.

Example: A 3D-cube with two weges deleted

Example: State space and trace space for a semaphore HDA



State space:
a 3D cube $\vec{I}^3 \setminus F$
minus 4 box obstructions
pairwise connected

Path space model contained in torus $(\partial\Delta^2)^2$ – homotopy equivalent to a wedge of two circles and a point: $(S^1 \vee S^1) \sqcup *$

Want to know more?

Tomorrow, 2:30 pm: Mini-symposium Applied and Computational Algebraic Topology

Some References

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Want to know more?
Thank you!

Books

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Related articles

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- Jardine, Path categories and resolutions, Homology, Homotopy Appl. **12** (2010), 231 – 244.
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Thank you for your attention!