Directed topology. An introduction

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Alfemøde, 8.5.2007

Outline

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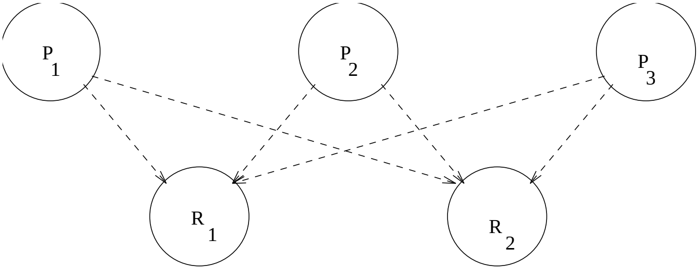
1. Motivations, mainly from Concurrency Theory
2. Directed topology: algebraic topology with a twist
3. A categorical framework (with examples)
4. "Compression" of ditopological categories: generalized congruences via homotopy flows

Main Collaborators:

- Lisbeth Fajstrup (Aalborg), Éric Goubault, Emmanuel Haucourt (CEA, France)

Motivation: Concurrency
Mutual exclusion

Mutual exclusion occurs, when n processes P_i compete for m resources R_j.



Only k processes can be served at any given time.

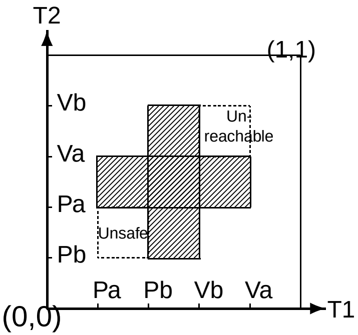
Semaphores!

Semantics: A processor has to lock a resource and relinquish the lock later on!

Description/abstraction $P_i : \dots PR_j \dots VR_j \dots$ (Dijkstra)

Schedules in ”progress graphs”

The Swiss flag example



PV-diagram from

$P_1 : P_a P_b V_b V_a$

$P_2 : P_b P_a V_a V_b$

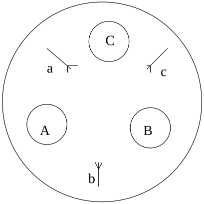
Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded).

Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions.

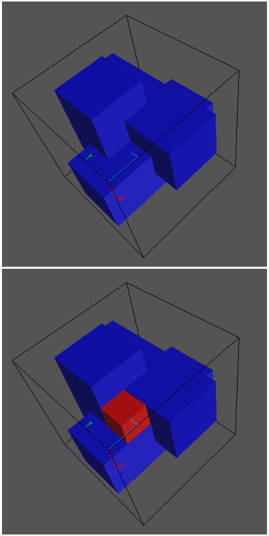
Deadlocks, unsafe and unreachable regions may occur.

Higher dimensional automata 1

Example: Dining philosophers; dimension 3 and beyond



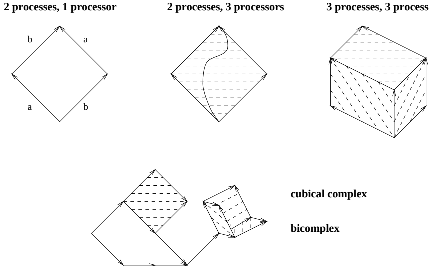
A=Pa.Pb.Va.Vb
B=Pb.Pc.Vb.Vc
C=Pc.Pa.Vc.Va

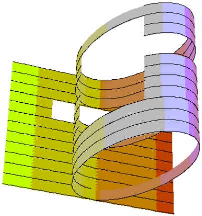


Higher dimensional complex with a forbidden region consisting of isothetic hypercubes and an unsafe region.

Higher dimensional automata 2
seen as (geometric realizations of) cubical sets

Vaughan Pratt, Rob van Glabbeek, Eric Goubault...





Squares/cubes/hypercubes are filled in iff actions on boundary are independent.

Higher dimensional automata are cubical sets:

- like simplicial sets, but modelled on (hyper)cubes instead of simplices; glueing by face maps (and degeneracies)
- additionally: preferred directions – not all paths allowable.

Discrete versus continuous models

How to handle the state-space explosion problem?

Discrete models for concurrency (transition graph models) suffer a severe problem if the number of processors and/or the length of programs grows: The number of states (and the number of possible schedules) grows exponentially: this is known as the state space explosion problem.

You need clever ways to find out which of the schedules yield equivalent results – e.g., to check for correctness – for general reasons.

Alternative: Infinite continuous models allowing for well-known equivalence relations on paths (homotopy = 1-parameter deformations) – but with an important twist!

Analogy: Continuous physics as an approximation to (discrete) quantum physics.

A framework for directed topology
d-spaces, M. Grandis (03)

X a topological space. $\vec{P}(X) \subseteq X^I = \{p : I = [0,1] \to X \text{ cont.}\}$ a set of d-paths ("directed" paths $\leftrightarrow$ executions) satisfying

- { constant paths } $\subseteq \vec{P}(X)$
- $\varphi \in \vec{P}(X)(x,y), \psi \in \vec{P}(X)(y,z) \Rightarrow \varphi * \psi \in \vec{P}(X)(x,z)$
- $\varphi \in \vec{P}(X), \alpha \in I^I$ a nondecreasing reparametrization $\Rightarrow \varphi \circ \alpha \in \vec{P}(X)$

The pair $(X, \vec{P}(X))$ is called a d-space.
Observe: $\vec{P}(X)$ is in general not closed under reversal:

$$\alpha(t) = 1 - t, \varphi \in \vec{P}(X) \not\Rightarrow \varphi \circ \alpha \in \vec{P}(X)!$$

Examples:

- An HDA with directed execution paths.
- A space-time(relativity) with time-like or causal curves.

Concepts from algebraic topology 1
Homotopy, fundamental group

basic: the category *Top* of topological spaces and continuous maps. $I = [0, 1]$ the unit interval.

Definition

- A continuous map $H : X \times I \to Y$ is called a homotopy.
- Continuous maps $f, g : X \to Y$ are called homotopic to each other if there is a homotopy H with $H(x, 0) = f(x), H(x, 1) = g(x), x \in X$.
- $[X, Y]$ the set of homotopy classes of continuous maps from X to Y.
- Variation: pointed continuous maps $f : (X, *) \to (Y, *)$ and pointed homotopies $H : (X \times I, * \times I) \to (Y, *)$.
- Loops in Y as the special case $X = S^1$ (unit circle).
- Fundamental group $\pi_1(Y, y) = [(S^1, *), (Y, y)]$ with product arising from concatenation and inverse from reversal.

d-maps, Dihomotopy, d-homotopy

A d-map $f : X \to Y$ is a continuous map satisfying

- $f(\vec{P}(X)) \subseteq \vec{P}(Y)$

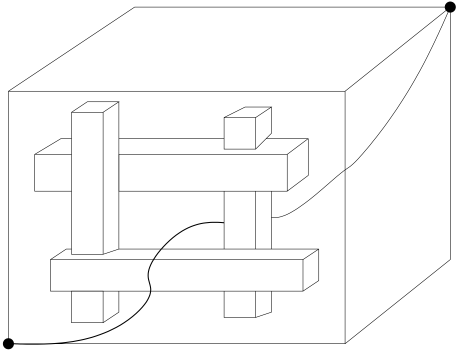
special case: $\vec{P}(I) = \{\sigma \in I^I | \sigma$ nondecreasing reparametrization$\}, \vec{I} = (I, \vec{P}(I))$.
Then $\vec{P}(X) =$ set of d-maps from $\vec{I}$ to X.

- Dihomotopy $H : X \times I \to Y$, every H_t a d-map
- elementary d-homotopy = d-map $H : X \times \vec{I} \to Y$ – $H_0 = f \xrightarrow{H} g = H_1$
- d-homotopy: symmetric and transitive closure (”zig-zag”)

L. Fajstrup, 05: In cubical models (for concurrency, e.g., HDAs), the two notions agree for d-paths ($X = \vec{I}$). In general, they do not.

Dihomotopy is finer than homotopy with fixed endpoints

Example: Two wedges in the forbidden region

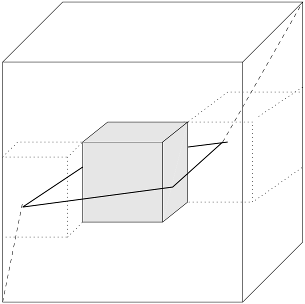


All dipaths from minimum to maximum are homotopic.
A dipath through the “hole” is not dihomotopic to a dipath on the boundary.

The twist has a price

Neither homogeneity nor cancellation nor group structure

In ordinary topology, it suffices to study loops in a space X with a given start=end point x (one per path component). Moreover: "Loops up to homotopy" $\rightsquigarrow$ fundamental group $\pi_1(X, x)$ – concatenation, inversion!



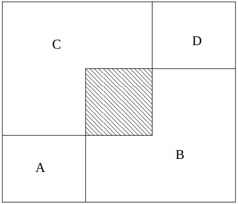
"Birth and death" of dihomotopy classes

Directed topology: Loops do not tell much; concatenation ok, cancellation not! Replace group structure by category structures!

A first remedy: the fundamental category

$\vec{\pi}_1(X)$ of a d-space X [Grandis:03, FGHR:04]:

- Objects: points in X
- Morphisms: d- or dihomotopy classes of d-paths in X
- Composition: from concatenation of d-paths



Property: van Kampen theorem (M. Grandis)
Drawbacks: Infinitely many objects. Calculations?
Question: How much does $\vec{\pi}_1(X)(x, y)$ depend on (x, y)?
Remedy: Localization, component category. [FGHR:04, GH:06]
Problem: “Compression” works only for loopfree categories

Concepts from algebraic topology 2 (for calculations)

Homotopy groups, homology groups, homotopy equivalences

- $\pi_n(X,x) = [(S^n,*),(X,x)]$; group structure: $S^n \to S^n \vee S^n$, abelian for $n > 1$. Easy to define, difficult to calculate.
- Homology and cohomology groups $H_n(X)$ and $H^n(X)$: abelian groups; definition more complicated, but essentially calculable for reasonable topological spaces. $H_0(X)$ free abelian group on path components of X. $H_1(X) = \pi_1(X)/_{[\pi_1(X),\pi_1(X)]}$.
- A continuous map $f:(X,x) \to (Y,y)$ induces group homomorphisms $f_\# : \pi_n(X,x) \to \pi_n(Y,y)$, and $f_* : H_n(X) \to H_n(Y),\ n \in \mathbf{N}$. Homotopic maps induce the same homomorphism (homotopy invariance). Functoriality: $(g \circ f)_\# = g_\# \circ f_\#, (g \circ f)_* = g_* \circ f_*$.
- A continuos map $f : X \to Y$ is a homotopy equivalence if there exists a homotopy inverse $g : Y \to X$ satisfying $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. Homotopy equivalent spaces have isomorphic homotopy and (co)homology groups.

Getting started: Traces – and trace categories

Get rid of (increasing) reparametrizations!

X a (saturated) d-space.
$\varphi, \psi \in \vec{P}(X)(x,y)$ are called reparametrization equivalent if there are $\alpha, \beta \in \vec{P}(I)$ such that $\varphi \circ \alpha = \psi \circ \beta$.
(Fahrenberg-R., JHRS**2**, 07): Reparametrization equivalence is an equivalence relation (transitivity).
$\vec{T}(X)(x,y) = \vec{P}(X)(x,y)/_{\simeq}$ makes $\vec{T}(X)$ into the (topologically enriched) trace category – composition associative!
A d-map $f : X \to Y$ induces a functor $\vec{T}(f) : \vec{T}(X) \to \vec{T}(Y)$.
Variant: $\vec{R}(X)(x,y)$ consists of regular d-paths (not constant on any non-trivial interval $J \subset I$). The contractible group $Homeo_+(I)$ of increasing homeomorphisms acts on these – freely if $x \neq y$.

Theorem (FR:JHRS2, 07)

$\vec{R}(X)(x,y)/_{\simeq} \to \vec{P}(X)(x,y)/_{\simeq}$ is a homeomorphism.

A d-structure on X induces the preorder $\preceq$:

$$x \preceq y \Leftrightarrow \vec{T}(X)(x,y) \neq \emptyset$$

and an indexing preorder category $\vec{D}(X)$ with

- Objects: pairs $(x,y), x \preceq y$
- Morphisms:
  $\vec{D}(X)((x,y),(x',y')) := \vec{T}(X)(x',x) \times \vec{T}(X)(y,y')$:

$$x' \rightrightarrows x \xrightarrow{\preceq} y \rightrightarrows y'$$

- Composition: by pairwise contra-, resp. covariant concatenation.

A d-map $f : X \to Y$ induces a functor $\vec{D}(f) : \vec{D}(X) \to \vec{D}(Y)$.

The trace space functor

Preorder categories organise the trace spaces

The preorder category organises X via the
trace space functor $\vec{T}^X : \vec{D}(X) \to Top$

- $\vec{T}^X(x,y) := \vec{T}(X)(x,y)$
- $\vec{T}^X(\sigma_x,\sigma_y):$ $\vec{T}(X)(x,y) \longrightarrow \vec{T}(X)(x',y')$

$$[\sigma] \longmapsto [\sigma_x * \sigma * \sigma_y]$$

Homotopical variant $\vec{D}_\pi(X)$ with morphisms
$\vec{D}_\pi(X)((x,y),(x',y')) := \vec{\pi}_1(X)(x',x) \times \vec{\pi}_1(X)(y,y')$
and trace space functor $\vec{T}^X_\pi : \vec{D}_\pi(X) \to Ho-Top$ (with homotopy classes as morphisms).

In less technical terms: Investigation of the d-path/trace spaces $\vec{T}(X)(x,y)$ on X with given endpoints x,y and the variation of their topology under change of endpoints.

Sensitivity with respect to variations of end points

A persistence point of view

- How much does (the homotopy type of) $\vec{T}^X(x, y)$ depend on (small) changes of x, y?
- Which concatenation maps $\vec{T}^X(\sigma_x, \sigma_y) : \vec{T}^X(x,y) \to \vec{T}^X(x',y'),\ [\sigma] \mapsto [\sigma_x * \sigma * \sigma_y]$ are homotopy equivalences, induce isos on homotopy, homology groups etc.?
- The persistence point of view: Homology classes etc. are born (at certain branchings/mergings) and may die (analogous to the framework of G. Carlsson etal.)
- Are there components with (homotopically/homologically) stable dipath spaces (between them)? Are there borders ("walls") at which changes occur?
- $\rightsquigarrow$ need a lot of bookkeeping!

Dihomology $\vec{H}_*$

- For every d-space X, there are homology functors

$$\vec{H}_{*+1}(X) = H_* \circ \vec{T}_{\pi}^X : \vec{D}_{\pi}(X) \to Ab,\ (x, y) \mapsto H_*(\vec{T}(X)(x, y))$$

  capturing homology of all relevant d-path spaces in X and the effects of the concatenation structure maps.
- A d-map $f : X \to Y$ induces a natural transformation $\vec{H}_{*+1}(f)$ from $\vec{H}_{*+1}(X)$ to $\vec{H}_{*+1}(Y)$.
- Properties? Calculations? Not much known in general. A master's student has studied this topic for X a cubical complex (its geometric realization) by constructing a cubical model for d-path spaces.
- Higher dihomotopy functors $\vec{\pi}_*$: in the same vain, a bit more complicated to define, since they have to reflect choices of base paths.

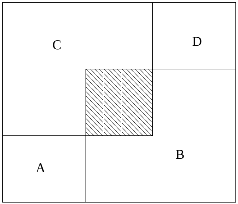
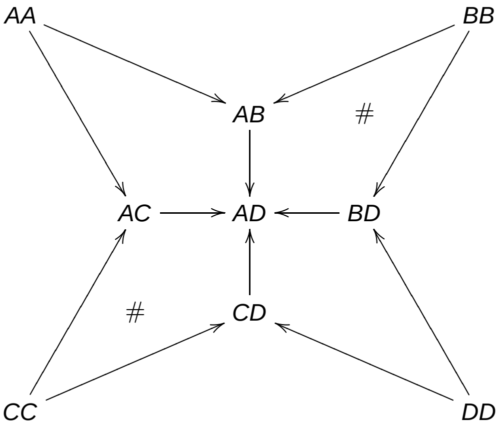


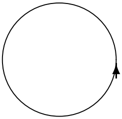
Figure: Standard example and component category



Components A, B, C, D – or rather $AA, AB, AC, AD, BB, BD, CC, CD, DD$.
$\#$: diagram commutes.

Examples of component categories
Oriented circle – with loops!

$X \quad = \quad \vec{S}^1$



oriented circle

$$\mathcal{C}: \Delta \underset{b}{\overset{a}{\rightleftarrows}} \bar{\Delta}$$

Δ the diagonal, $\bar{\Delta}$ its complement.
$\mathcal{C}$ is the free category generated by a, b.

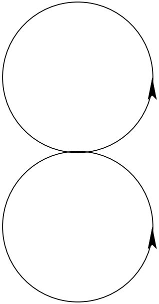
- Remark that the components are no longer products!
- It is essential in order to get a discrete component category to use an indexing category taking care of pairs (source, target).

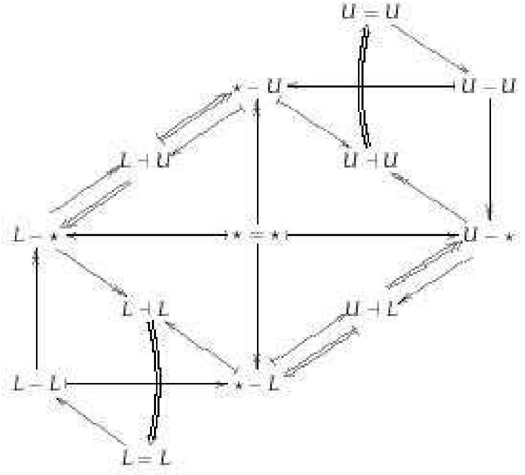
Component categories
via generalized congruences and homotopy flows

- How to identify morphisms in a category between different objects in an organised manner? Localization or Generalized congruence (Bednarczyk, Borzyszkowski, Pawlowski, TAC 1999) $\rightsquigarrow$ quotient category with identifications on both objects and morphisms.
- Homotopy flows (MR, ACS 2007) identify both elements and d-paths: Like flows in differential geometry. Instead of diffeotopies: Self-homotopies inducing homotopy equivalences on spaces of d-paths with given end points ("automorphic").
- Automorphic homotopy flows give rise to significant generalized congruences. Corresponding component category $\vec{D}_\pi(X)/\simeq$ identifies pairs of points on the same "homotopy flow line" and (chains of) morphisms.

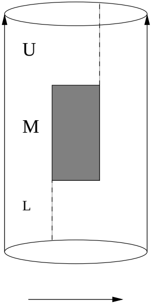
The component category of a wedge of two oriented circles

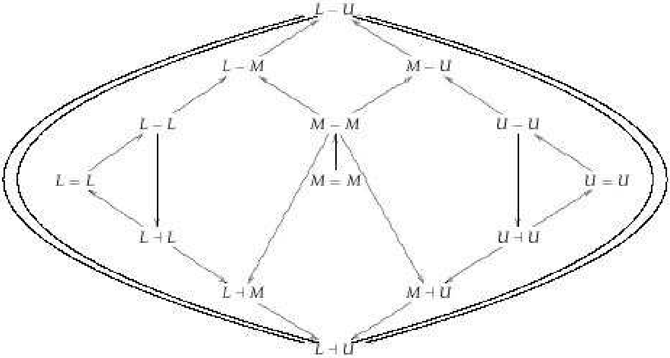
$X = \vec{S}^1 \vee \vec{S}^1$





The component category of an oriented cylinder with a deleted rectangle





Concluding remarks

- Component categories contain the essential information given by (algebraic topological invariants of) d-path spaces
- Compression via component categories is an antidote to the state space explosion problem
- Some of the ideas (for the fundamental category) are implemented and have been tested for huge industrial software from EDF (Éric Goubault & Co., CEA)
- Dihomotopy equivalence: Definition uses automorphic homotopy flows to ensure homotopy equivalences

$$\vec{T}(f)(x,y) : \vec{T}(X)(x,y) \to \vec{T}(Y)(fx,fy) \text{ for all } x \preceq y.$$

- Much more theoretical and practical work remains to be done!