Simplicial models for trace spaces

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FIELDS

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Examples: State spaces and associated path spaces in Higher Dimensional Automata (HDA)

Motivation: Concurrency

Simplest case: State spaces and path spaces related to linear PV-programs

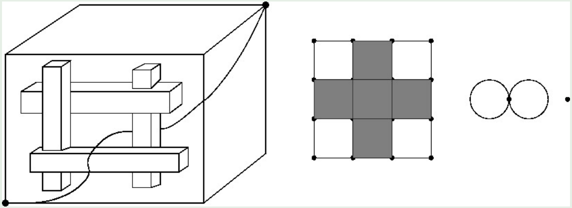
Tool: Cutting up path spaces into contractible subspaces

Homotopy type of path space described by a matrix poset category and realized by a prodsimplicial complex

Algorithmics: Detecting dead and alive subcomplexes/matrices

Outlook: How to handle general HDA

Example 1: State space and trace space for a semaphore HDA



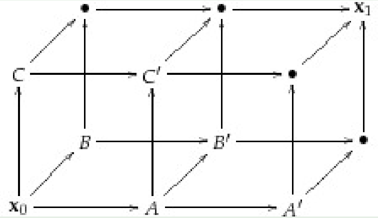
State space:
a 3D cube $\vec{I}^3 \setminus F$
minus 4 box obstructions

Path space model contained in torus $(\partial\Delta^2)^2$ – homotopy equivalent to a wedge of two circles and a point: $(S^1 \vee S^1) \sqcup *$

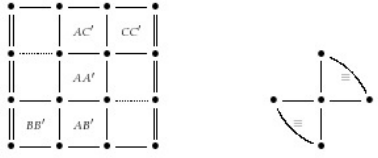
Analogy in standard algebraic topology

Relation between space X and loop space ΩX.

Example 2: State space and trace space for a non-looping pre-cubical complex

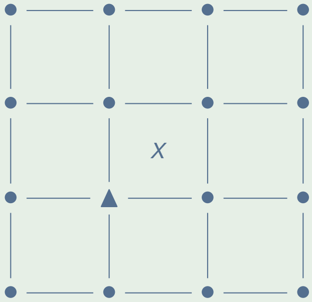


State space: Boundaries of two cubes glued together at common square $AB'C'\bullet$



Path space model: Prodsimplicial complex contained in $(\partial\Delta^2)^2 \cup \partial\Delta^2$– homotopy equivalent to $S^1 \vee S^1$

Example 3: Torus with a hole

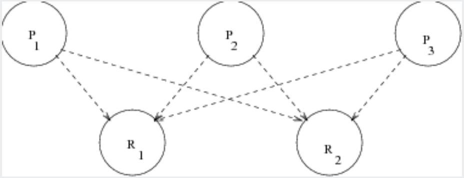


Path space model:
Discrete infinite space of dimension 0 corresponding to $\{r, u\}^*$

State space with hole X:
2D torus $\partial\Delta^2 \times \partial\Delta^2$ with a rectangle $\Delta^1 \times \Delta^1$ removed

Mutual exclusion

occurs, when n processes P_i compete for m resources R_j.



Only k processes can be served at any given time.

Semaphores

Semantics: A processor has to lock a resource and to relinquish the lock later on!
Description/abstraction: $P_i : \ldots PR_j \ldots VR_j \ldots$ (E.W. Dijkstra)
P: probeer; V: verhoog

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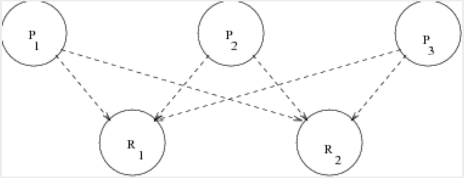

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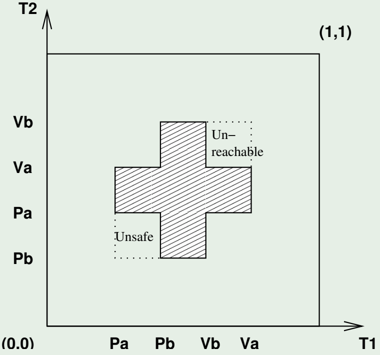
A geometric model: Schedules in "progress graphs"

The Swiss flag example



PV-diagram from

$P_1 : P_a P_b V_b V_a$

$P_2 : P_b P_a V_a V_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded). Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions. Deadlocks, unsafe and unreachable regions may occur.

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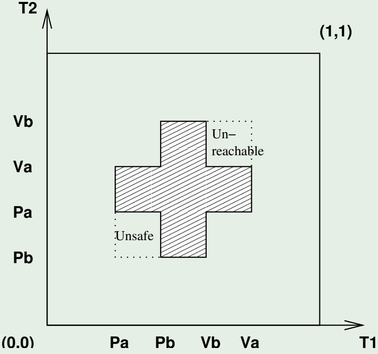

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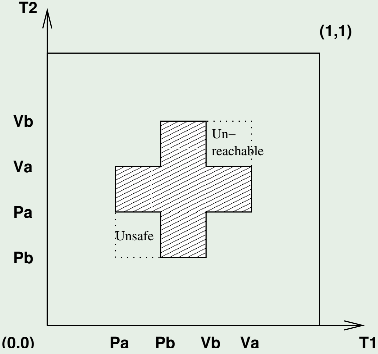

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The state space

A linear PV-program is modeled as the complement of a forbidden region F consisting of a number of holes in an n-cube:
Hole = isothetic hyperrectangle
$R^i =]a_1^i, b_1^i[\times \cdots \times]a_n^i, b_n^i[\subset I^n, 1 \leq i \leq l$:
with minimal vertex $\mathbf{a}^i$ and maximal vertex $\mathbf{b}^i$.
State space $X = \vec{I}^n \setminus F, \; F = \bigcup_{i=1}^l R^i$
X inherits a partial order from $\vec{I}^n$.

More general (PV)-programs:

- Replace $\vec{I}^n$ by a product $\Gamma_1 \times \cdots \times \Gamma_n$ of digraphs.
- Holes have then the form $p_1^i((0,1)) \times \cdots \times p_n^i((0,1))$ with $p_j^i : \vec{I} \to \Gamma_j$ a directed injective (d-)path.
- Pre-cubical complexes: like pre-simplicial complexes, with (partially ordered) hypercubes instead of simplices as building blocks.

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Definition

- X a d-space, $a, b \in X$.
  $p : \vec{I} \to X$ a d-path in X (continuous and "order-preserving") from a to b.
- $\vec{P}(X)(a,b) = \{p : \vec{I} \to X \mid p(0) = a, p(b) = 1, p \text{ a d-path}\}$.
  Trace space $\vec{T}(X)(a,b) = \vec{P}(X)(a,b)$ modulo increasing reparametrizations.
  In most cases: $\vec{P}(X)(a,b) \simeq \vec{T}(X)(a,b)$.
- A dihomotopy on $\vec{P}(X)(a,b)$ is a map $H : \vec{I} \times I \to X$ such that $H_t \in \vec{P}(X)(a,b),\ t \in I$; ie a path in $\vec{P}(X)(a,b)$.

Aim:

Description of the homotopy type of $\vec{P}(X)(a,b)$ as explicit finite dimensional prodsimplicial complex.
In particular: its path components, ie the dihomotopy classes of d-paths (executions).

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Tool: Subspaces of X and of $\vec{P}(X)(\mathbf{0},\mathbf{1})$

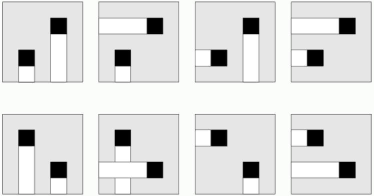
$X = \vec{I}^n \setminus F,\ F = \bigcup_{i=1}^{l} R^i;\ R^i = [\mathbf{a}^i, \mathbf{b}^i];\ \mathbf{0}, \mathbf{1}$ the two corners in I^n.

Definition

1. $X_{ij} = \{x \in X \mid x \leq \mathbf{b}^i \Rightarrow x_j \leq a^i_j\}$ – direction j restricted at hole i
2. M a binary $l \times n$-matrix: $X_M = \bigcap_{m_{ij}=1} X_{ij}$

First Examples:

$M =$ $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$



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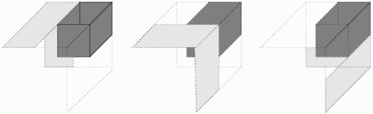

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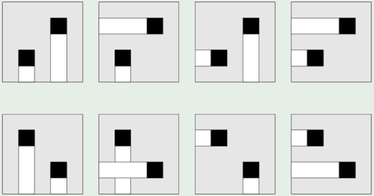

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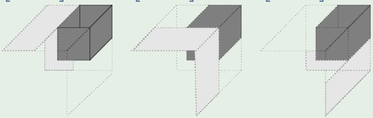

Binary matrices

$M_{l,n}$ poset ($\leq$) of binary $l \times n$-matrices

$M_{l,n}^{R,*}$ no row vector is the zero vector

$M_{l,n}^{R,u}$ every row vector is a unit vector

$M_{l,n}^{C,u}$ every column vector is a unit vector

A cover:

$$\vec{P}(X)(\mathbf{0},\mathbf{1}) = \bigcup_{M \in M_{l,n}^{R,u}} \vec{P}(X_M)(\mathbf{0},\mathbf{1}).$$

Theorem

Every path space $\vec{P}(X_M)(\mathbf{0},\mathbf{1})$, $M \in M_{l,n}^{R,*}$, is empty or contractible. *Which is which?*

Proof.

Subspaces X_M, $M \in M_{l,n}^{R,*}$ are closed under $\vee$ = l.u.b. □

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Covers by contractible (or empty) subspaces

Bookkeeping with binary matrices

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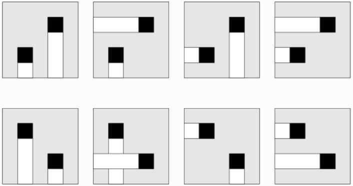
$M \in \mathcal{C}(X)(\mathbf{0},\mathbf{1})$ "alive"

Topology:
prodsimplicial complex
$\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \subseteq (\Delta^{n-1})^l$
$\Delta_M = \Delta_{m_1} \times \cdots \times \Delta_{m_l} \subseteq \mathbf{T}(X)(\mathbf{0},\mathbf{1})$

$\Leftrightarrow \vec{P}(X_M)(\mathbf{0},\mathbf{1}) \neq \emptyset.$

Examples of path spaces



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A combinatorial model and its geometric realization

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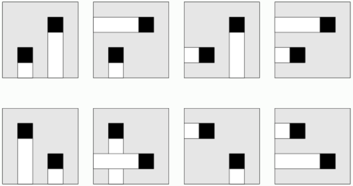

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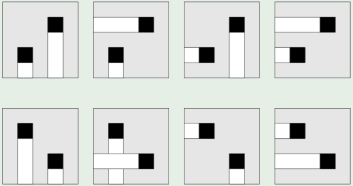

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Further examples

State spaces, “alive” matrices and path spaces

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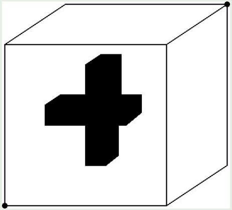
2. 

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   - $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M_{1,n}^{R,*} \setminus \{[1,\dots 1]\}$.
   - $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) = \partial\Delta^{n-1} \simeq S^{n-2}$.
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   - $\mathcal{C}_{max}(X)(\mathbf{0},\mathbf{1}) = \left\{ \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \right\}$.
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More examples in Mimram’s talk!

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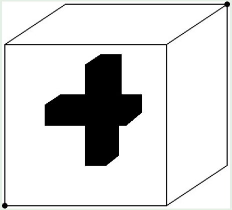
2. 

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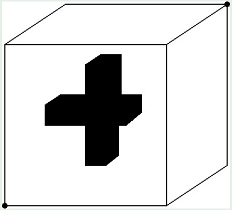
2. 

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   - $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M_{1,n}^{R,*} \setminus \{[1,\dots 1]\}$.
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   - $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) =$ 3 diagonal squares $\subset (\partial\Delta^2)^2 = T^2 \simeq S^1$.

More examples in Mimram’s talk!

Homotopy equivalence between trace space $\vec{T}(X)(\mathbf{0},\mathbf{1})$ and the prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

Theorem (A variant of the nerve lemma)

$\vec{P}(X)(\mathbf{0},\mathbf{1}) \simeq \mathbf{T}(X)(\mathbf{0},\mathbf{1}) \simeq \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.

Proof.

- Functors $\mathcal{D},\mathcal{E},\mathcal{T} : \mathcal{C}(X)(\mathbf{0},\mathbf{1})^{(op)} \to \mathbf{Top}$:
  $\mathcal{D}(M) = \vec{P}(X_M)(\mathbf{0},\mathbf{1})$,
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- colim $\mathcal{D} = \vec{P}(X)(\mathbf{0},\mathbf{1})$, colim $\mathcal{E} = \mathbf{T}(X)(\mathbf{0},\mathbf{1})$,
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- The trivial natural transformations $\mathcal{D} \Rightarrow \mathcal{T}, \mathcal{E} \Rightarrow \mathcal{T}$ yield:
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- Projection lemma:
  hocolim $\mathcal{D} \simeq$ colim $\mathcal{D}$, hocolim $\mathcal{E} \simeq$ colim $\mathcal{E}$.

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Why prodsimplicial?
rather than simplicial

- We distinguish, for every obstruction, sets $J_i \subset [1:n]$ of restrictions. A joint restriction is of product type $J_1 \times \cdots \times J_l \subset [1:n]^l$, and not an arbitrary subset of $[1:n]^l$.
- Simplicial model: a subcomplex of Δ^{n^l} – $2^{(n^l)}$ subsimplices.
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From $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ to properties of path space

Questions answered by homology calculations using $\mathbf{T}(X)(\mathbf{0}, \mathbf{1})$

Questions

- Is $\vec{P}(X)(\mathbf{0}, \mathbf{1})$ path-connected, i.e., are all (execution) d-paths dihomotopic (lead to the same result)?
- Determination of path-components?
- Are components simply connected?
  Other topological properties?

Strategies – Attempts

- Implementation of $\mathbf{T}(X)(\mathbf{0}, \mathbf{1})$ in ALCOOL:
  Progress at CEA/LIX-lab.: Goubault, Haucourt, Mimram
- The prodsimplicial structure on $\mathcal{C}(X)(\mathbf{0}, \mathbf{1}) \leftrightarrow \mathbf{T}(X)(\mathbf{0}, \mathbf{1})$ leads to an associated chain complex of vector spaces over a field.
- Use fast algorithms (eg Mrozek CrHom etc) to calculate the homology groups of these chain complexes even for very big complexes: M. Juda (Krakow).
- Number of path-components: $rkH_0(\mathbf{T}(X)(\mathbf{0}, \mathbf{1}))$.
  For path-components alone, there are faster “discrete” methods, that also yield representatives in each path component: Mimram’s talk!

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Allow less = forbid more!

Remove **extended** hyperrectangles R_j^i
$:= [0, b_1^i[\times \cdots \times [0, b_{j-1}^i[\times]a_j^i, b_j^i[\times [0, b_{j+1}^i[\times \cdots \times [0, b_n^i[\supset R^i.$

$$X_M = X \setminus \bigcup_{m_{ij}=1} R_j^i.$$

Theorem

The following are equivalent:

1. $\vec{P}(X_M)(\mathbf{0}, \mathbf{1}) = \emptyset \Leftrightarrow M \notin \mathcal{C}(X)(\mathbf{0}, \mathbf{1})$.
2. *There is a "dead" matrix* $N \leq M$, $N \in M_{l,n}^{C,u}$ *such that* $\bigcap_{n_{ij}=1} R_j^i \neq \emptyset$ *– giving rise to a deadlock unavoidable from* $\mathbf{0}$, *i.e.,* $T(X_N)(\mathbf{0}, \mathbf{1}) = \emptyset$.

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Dead matrices in $D(X)(\mathbf{0},\mathbf{1})$

Inequalities decide

Decisions: Inequalities

Deadlock algorithm (Fajstrup, Goubault, Raussen) $\rightsquigarrow$:

Theorem

- $N \in M^{C,u}_{l,n}$ *dead* $\Leftrightarrow$
  For all $1 \le j \le n$, *for all* $1 \le k \le n$ *such that* $\exists j' : n_{kj'} = 1$:

$$n_{ij} = 1 \Rightarrow a^i_j < b^k_j.$$

- $M \in M^{R,*}_{l,n}$ *dead* $\Leftrightarrow \exists N \in M^{C,u}_{l,n}$ *dead*, $N \le M$.

Definition

$D(X)(\mathbf{0},\mathbf{1}) := \{P \in M_{l,n} | \exists N \in M^{C,u}_{l,n}, N \textit{ dead} : N \le P\}$.

A cube with a cube hole

- $X = \vec{I}^n \setminus \vec{J}^n$
- $D(X)(\mathbf{0},\mathbf{1}) = \{[1,\ldots,1]\} = M^{C,u}_{1,n}$.

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Maximal alive $\leftrightarrow$ minimal dead

Still alive – not yet dead

- $\mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1}) \subset \mathcal{C}(X)(\mathbf{0},\mathbf{1})$ maximal alive matrices.
- Matrices in $\mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1})$ correspond to maximal simplex products in $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$.
- Connection: $M \in \mathcal{C}_{\max}(X)(\mathbf{0},\mathbf{1})$, $M \leq N$ a succesor (a single 0 replaced by a 1) $\Rightarrow N \in D(X)(\mathbf{0},\mathbf{1})$.

A cube removed from a cube

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- $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M^R_{l,n} \setminus \{[1,\dots,1]\}$;
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More general linear semaphore state spaces

- More general semaphores (intersection with the boundary $\partial I^n \subset I^n$ allowed)
- n dining philosophers: Trace space has $2^n - 2$ contractible components!
- Different end points: $\vec{P}(X)(\mathbf{c}, \mathbf{d})$ and iterative calculations
- End complexes rather than end points (allowing processes not to respond..., Herlihy & Cie)

State space components

New light on definition and determination of components of model space X.

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2a. Semaphores corresponding to non-linear programs:

Path spaces in product of digraphs

Products of digraphs instead of $\vec{I}^n$:
$\Gamma = \prod_{j=1}^n \Gamma_j$, state space $X = \Gamma \setminus F$,
F a product of generalized hyperrectangles R^i.

- $\vec{P}(\Gamma)(\mathbf{x}, \mathbf{y}) = \prod \vec{P}(\Gamma_j)(x_j, y_j)$ – homotopy discrete!

Pullback to linear situation

Represent a path component $C \in \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$ by (regular) d-paths $p_j \in \vec{P}(\Gamma_j)(x_j, y_j)$ – an interleaving.
The map $c : \vec{I}^n \to \Gamma, c(t_1, \ldots, t_n) = (c_1(t_1), \ldots, c_n(t_n))$ induces a homeomorphism $\circ c : \vec{P}(\vec{I}^n)(\mathbf{0}, \mathbf{1}) \to C \subset \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$.

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Homotopy types of interleaving components

Pull back F via c:
$\bar{X} = \vec{I}^n \setminus \bar{F}, \bar{F} = \bigcup \bar{R}^i, \bar{R}^i = c^{-1}(R^i)$ – honest hyperrectangles!
$i_X : \vec{P}(X) \hookrightarrow \vec{P}(\Gamma)$.
Given a component $C \subset \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$.
The d-map $c : \bar{X} \to X$ induces a homeomorphism
$c\circ : \vec{P}(\bar{X})(\mathbf{0}, \mathbf{1}) \to i_X^{-1}(C) \subset \vec{P}(\Gamma)(\mathbf{x}, \mathbf{y})$.

- C "lifts to X" $\Leftrightarrow \vec{P}(\bar{X})(\mathbf{0}, \mathbf{1}) \neq \emptyset$; if so:
- Analyse $i_X^{-1}(C)$ via $\vec{P}(\bar{X})(\mathbf{0}, \mathbf{1})$.
- Exploit relations between various components.

Special case: $\Gamma = (S^1)^n$ – a torus

State space: A torus with rectangular holes in F:
Investigated by Fajstrup, Goubault, Mimram etal.:
Analyse by language on the alphabet $\mathcal{C}(X)(\mathbf{0}, \mathbf{1})$ of alive matrices for a one-fold delooping of $\Gamma \setminus F$.

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HDA: Directed pre-cubical complex

Higher Dimensional Automaton: Pre-cubical complex – like simplicial complex but with cubes as building blocks – with preferred diretions.
Geometric realization X with d-space structure.

Branch points and branch cubes

These complexes have branch points and branch cells – more than one maximal cell with same lower corner vertex.
At branch points, one can cut up a cubical complex into simpler pieces.
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Non-branching complexes

Start with complex **without directed loops**: After finally many iterations: Subcomplex Y **without branch points**.

Theorem

$\vec{P}(Y)(\mathbf{x}_0, \mathbf{x}_1)$ is empty or contractible.

Proof.

Such a subcomplex has a preferred **diagonal flow** and a contraction from path space to the flow line from start to end. □

Branch category

Results in a (complicated) finite **branch category** $\mathcal{M}(X)(\mathbf{x}_0, \mathbf{x}_1)$ on subsets of set of (iterated) branch cells.

Theorem

$\vec{P}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is homotopy equivalent to the nerve $\mathcal{N}(\mathcal{M}(X)(\mathbf{x}_0, \mathbf{x}_1))$ of that category.

Extensions

3b. Path spaces for HDAs without d-loops

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Delooping HDAs

A pre-cubical complex comes with an L_1-length 1-form ω reducing to $\omega = dx_1 + \cdots + dx_n$ on every n-cube.
Integration: L_1-length on rectifiable paths, homotopy invariant.
Defines $l : P(X)(x_0, x_1) \to \mathbf{R}$ and $l_\sharp : \pi_1(X) \to \mathbf{R}$ with kernel K.
The (usual) covering $\tilde{X} \downarrow X$ with $\pi_1(\tilde{X}) = K$ is a directed pre-cubical complex without d- loops.

Theorem (Decomposition theorem)

For every pair of points $\mathbf{x}_0, \mathbf{x}_1 \in X$, path space $\vec{P}(X)(\mathbf{x}_0, \mathbf{x}_1)$ is homeomorphic to the disjoint union $\bigsqcup_{n\in\mathbf{Z}} \vec{P}(\tilde{X})(\mathbf{x}_0^0, \mathbf{x}_1^n)$[a].

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To conclude

- From a (rather compact) state space model to a finite dimensional trace space model.
- Calculations of invariants (Betti numbers) of path space possible even for quite large state spaces.
- Dimension of trace space model reflects not the size but the complexity of state space (number of obstructions, number of processors) – linearly.
- Challenge: General properties of path spaces for algorithms solving types of problems in a distributed manner?
  (Connection to the work of Herlihy and Rajsbaum)

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Want to know more?

Thank you!

- Samuel Mimram's subsequent talk!

References

- MR, Simplicial models for trace spaces, AGT **10** (2010), 1683 – 1714.
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Thank you for your attention!

- Samuel Mimram's subsequent talk!

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