Catenary*

Keywords: Shape of transmission lines, plane curves, arch length, first order separable differential equations, hyperbolic functions

Figure 1: Transmission lines crossing Limfjorden near Nordjyllandsværket.

The problem we want to deal with: What is the shape of a transmission line?

To make things simple we assume that a transmission line acts as a homogeneous flexible inextensible string hanging from two points only under the influence of gravity.

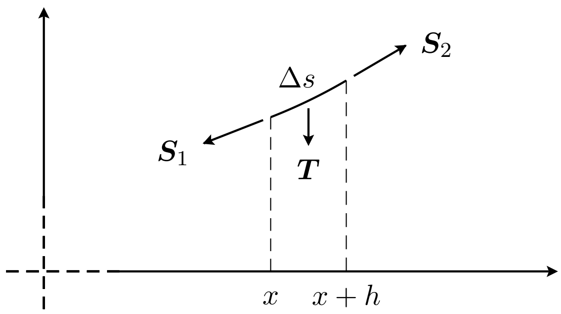

Figure 2: String forces and gravity acting on a part of the string.

$$\boldsymbol{S}_1 = -k_1(\boldsymbol{i} + f'(x)\boldsymbol{j})$$
$$\boldsymbol{S}_2 = k_2(\boldsymbol{i} + f'(x+h)\boldsymbol{j})$$
$$\boldsymbol{T} = -(\rho\Delta s)g\boldsymbol{k}$$

ρ is the density per unit length
g is the gravity acceleration

Note that the string forces acts in direction of the curve tangents.

Projection on the x-axis gives

$$-k_1 + k_2 = 0 \quad \Rightarrow \quad k_1 = k_2 = k \quad \text{for all } x,$$

*In Danish: Kædelinie

and projection on the y-axis gives

$$-kf'(x) + kf'(x+h) = -\rho g \Delta s \quad \Rightarrow \quad \frac{f'(x+h) - f'(x)}{h} = \frac{\rho g}{k} \frac{\Delta s}{h}.$$

Letting $h \to 0$ in the last equation leads to

$$f''(x) = \frac{1}{a} \frac{ds}{dx}, \quad a = \frac{k}{\rho g},$$

where ds is the arch length element. Substituting $\sqrt{1 + f'(x)^2}\, dx$ for ds gives

$$f''(x) = \frac{1}{a} \sqrt{1 + f'(x)^2},$$

a second order differential equation. By setting $u(x) = f'(x)$, the second order equation can be split into two first order differential equations

$$u'(x) = \frac{1}{a} \sqrt{1 + u(x)^2} \quad \text{and} \quad f'(x) = u(x).$$

The first equation in a slightly different notation

$$\frac{du}{dx} = \frac{1}{a} \sqrt{1 + u^2}$$

can be recognised as a separable equation, hence

$$\frac{du}{\sqrt{1 + u^2}} = \frac{1}{a} dx.$$

To solve this equation we need a bit of knowledge of hyperbolic functions. These functions have been introduced in 'Kompendium i lineær algebra' to illustrate some theoretical aspects of linear algebra, see pages 4, 5, 28, 29, and 31.

Here we will make use of the fundamental relation between the hyperbolic cosine and the hyperbolic sine

$$\cosh^2 x - \sinh^2 x = 1,$$

which is easily proven from the definitions of cosh and sinh. Regarding differentiation of cosh and sinh see 'Kompendium i lineær algebra' page 29.

Going back to the separated equation we introduce the substitution

$$u = \sinh t \quad \Rightarrow \quad du = \cosh t \; dt$$

on the left side. The inverse substitution is

$$t = \operatorname{arsinh} u.$$

(The inverse functions to cosh and sinh are called area functions.) Now

$$u = \sinh t \quad \Rightarrow \quad \sqrt{1 + u^2} = \sqrt{1 + \sinh^2 t} = \sqrt{\cosh^2 t} = \cosh t,$$

hence

$$\frac{du}{\sqrt{1 + u^2}} = \frac{\cosh t \; dt}{\cosh t} = dt,$$

and the solution of the differential equation follows as

$$t = \frac{1}{a} x + c \quad \Rightarrow \quad \operatorname{arsinh} u = \frac{1}{a} x + c \quad \Rightarrow \quad u(x) = \sinh(\frac{1}{a} x + c).$$

Using the initial condition $u(0) = 0$, the arbitrary constant c becomes 0. Thus

$$u(x) = \sinh \frac{x}{a}.$$

To solve the second differential equation $f'(x) = u(x)$ we only need to find an antiderivative to $u(x)$:

$$f(x) = \int \sinh \frac{x}{a} \, dx = a \int \sinh \frac{x}{a} \, d(\frac{x}{a}) = a \cosh \frac{x}{a} + c$$

Using the initial condition $f(0) = a$ we get $c = 0$.

Finally we can conclude that the shape of a transmission line is identical with that of a hyperbolic cosine

$$f(x) = a \cosh \frac{x}{a},$$

which in this context is called a catenary (Latin *catena* chain).

Note that this result is valid for arbitrary choices of supporting points, i.e., with masts in different heights, the transmission lines between masts will always form catenaries.

Appendix

Looking up the antiderivative

$$\int \frac{1}{\sqrt{x^2+1}} dx$$

in a table of integrals we find

$$\ln(x + \sqrt{x^2+1}).$$

Anything wrong in the calculation above? To answer that let us find the inverse function to the hyperbolic sine:

$$y = \sinh x \quad \Leftrightarrow \quad y = \frac{e^x - e^{-x}}{2} = \frac{e^{2x} - 1}{2e^x} \quad \Leftrightarrow \quad e^{2x} - 2ye^x - 1 = 0$$

The last equation is a quadratic equation in e^x, which we can solve:

$$e^x = \frac{2y \overset{+}{(-)} \sqrt{4y^2+4}}{2} = y \overset{+}{(-)} \sqrt{y^2+1} \quad \Rightarrow \quad x = \ln(y + \sqrt{y^2+1})$$

Interchanging the variables x and y we obtain the identity

$$\operatorname{arsinh} x = \ln(x + \sqrt{x^2+1}).$$

Further reading:

Edwards & Penney, Calculus Early Transcendentals, 7th ed., Section 6.9 and Section 8.3.

Wikipedia, `http://en.wikipedia.org/wiki/Catenary`

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