2

5 $z_1 z_2 = (x_1 + i y_1)(x_2 + i y_2) = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$

$= x_2 x_1 - y_2 y_1 + i(x_2 y_1 + x_1 y_2) = z_2 z_1$ dvs. kommutativ

11 $z^2 + z + 1 = 0, \quad z = (x, y)$

$(x,y)^2 + (x,y) + (1,0) = (x^2 - y^2, 2xy) + (x,y) + (1,0) = (x^2 - y^2 + x + 1, 2xy + y) = (0,0)$

$\Rightarrow x^2 - y^2 + x + 1 = 0 \wedge 2xy + y = 0 \Rightarrow x^2 - y^2 + x + 1 = 0 \wedge (y = 0 \vee x = -\frac{1}{2})$

$\Rightarrow (x^2 + x + 1 = 0 \wedge y = 0) \vee (y^2 = \frac{3}{4} \wedge x = -\frac{1}{2}) \Rightarrow (x, y) = (-\frac{1}{2}, \pm \frac{\sqrt{3}}{2})$

3

1 a $\quad \dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i} = \dfrac{(1+2i)(3+4i)}{9+16} - \dfrac{1+2i}{5} = \dfrac{-5+10i - 5 - 10i}{25} = -\dfrac{2}{5}$

b $\quad \dfrac{5i}{(1-i)(2-i)(3-i)} = \dfrac{5i}{(1-3i)(3-i)} = \dfrac{5i}{-10i} = -\dfrac{1}{2}$

c $\quad (1-i)^4 = (-2i)^2 = -4$

8 Binomialformeln: $(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}$

Induktionsbevis (a) $n = 0$: $z_1 + z_2 = z_1 + z_2$ ok

(b) gäller för $n = m$, sätt $n = m+1$:

$$(z_1 + z_2)^{m+1} = (z_1 + z_2)(z_1 + z_2)^m = (z_1 + z_2) \sum_{k=0}^{m} \binom{m}{k} z_1^k z_2^{m-k}$$

$$= \sum_{k=0}^{m} \binom{m}{k} z_1^{k+1} z_2^{m-k} + \sum_{k=0}^{m} \binom{m}{k} z_1^k z_2^{m+1-k}$$

$$= \sum_{k=1}^{m+1} \binom{m}{k-1} z_1^k z_2^{m+1-k} + \sum_{k=0}^{m} \binom{m}{k} z_1^k z_2^{m+1-k}$$

$$= \sum_{k=1}^{m} \binom{m}{k-1} z_1^k z_2^{m+1-k} + z_1^{m+1} + z_2^{m+1} + \sum_{k=1}^{m} \binom{m}{k} z_1^k z_2^{m+1-k}$$

$$= z_1^{m+1} + \sum_{k=1}^{m} \left(\binom{m}{k-1} + \binom{m}{k} \right) z_1^k z_2^{m+1-k} + z_2^{m+1}$$

$$= z_1^{m+1} + \sum_{k=1}^{m} \binom{m+1}{k} z_1^k z_2^{m+1-k} + z_2^{m+1}, \quad \text{jfr. [illegible] till sand.}$$

$$= \sum_{k=0}^{m+1} \binom{m+1}{k} z_1^k z_2^{m+1-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2^{n-k}$$

4

3 $\quad \dfrac{\mathrm{Re}(z_1 + z_2)}{|z_3 + z_4|} \le \dfrac{|z_1 + z_2|}{||z_3| - |z_4||} \le \dfrac{|z_1| + |z_2|}{||z_3| - |z_4||} \quad (|z_3| \neq |z_4|)$

4

6 a $|z-4i|+|z+4i|=10$ er ellipse med brennpunkter i $(0,4)$ og $(0,-4)$ ifølge def. på ellipse

Ligning:

$\sqrt{x^2+(y-4)^2}+\sqrt{x^2+(y+4)^2}=10$

$x^2+y^2-8y+16+x^2+y^2+8y+16+2\sqrt{(x^2+y^2-8y+16)(x^2+y^2+8y+16)}=100$

$(x^2+y^2+16)^2-64y^2=(50-x^2-y^2-16)^2$

$100x^2+36y^2=900$

$\frac{x^2}{3^2}+\frac{y^2}{5^2}=1$; $1-e^2=\frac{9}{25} \Rightarrow e=\frac{4}{5} \Rightarrow 5e=5\cdot\frac{4}{5}=4$ ok

b $|z-1|=|z+i| \Leftrightarrow (x-1)^2+y^2=x^2+(y+1)^2 \Rightarrow y=-x$

5

3 i $\overline{(x_1+iy_1)-(x_2+iy_2)}=\overline{x_1-x_2+i(y_1-y_2)}=x_1-x_2-i(y_1-y_2)$

$=x_1-iy_1-(x_2-iy_2)=\bar{z}_1-\bar{z}_2$

ii $\overline{z_1z_2}=\overline{x_1x_2-y_1y_2+i(x_1y_2+x_2y_1)}=x_1x_2-y_1y_2-i(x_1y_2+x_2y_1)$

$=(x_1-iy_1)(x_2-iy_2)=\bar{z}_1\bar{z}_2$

7 $|\mathrm{Re}(z+\bar{z}+z^3)|\le|z+\bar{z}+z^3|\le 2+|z|+|z|^3\le 2+1+1^3=4$

for $|z|\le 1$

8

1 a $z=\frac{i}{-2-2i}$, $\mathrm{Arg}\,z=\frac{\pi}{2}-\frac{3\pi}{4}=-\frac{\pi}{4}$

b $z=(\sqrt{3}-i)^6$, $\arg z=\left(-\frac{\pi}{6}\right)^6=-\pi$, $\mathrm{Arg}\,z=\pi$

9 i $(1-z)(1+z+\dots+z^n)=1+z+\dots+z^n-z-z^2-\dots-z^{n+1}=1-z^{n+1}$

$\Rightarrow 1+z+z^2+\dots+z^n=\frac{1-z^{n+1}}{1-z}$, $z\neq 1$

ii $z=e^{i\theta}$:

$\mathrm{Re}\,(1+e^{i\theta}+e^{i2\theta}+\dots+e^{in\theta})=1+\cos\theta+\cos 2\theta+\cos n\theta$

$\mathrm{Re}\,\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}=\mathrm{Re}\,\frac{(1+e^{i\theta})(1-e^{i(n+1)\theta})}{1-e^{2i\theta}}$

$=\mathrm{Re}\,\frac{(1-\cos\theta+i\sin\theta)(1-\cos(n+1)\theta-i\sin(n+1)\theta)}{(1-\cos\theta)^2+\sin^2\theta}$

$=\frac{1-\cos\theta}{2(1-\cos\theta)}+\frac{-(1-\cos\theta)\cos(n+1)\theta+\sin\theta\sin(n+1)\theta}{}$

$=\frac{1}{2}+\frac{-2\sin^2\frac{\theta}{2}\cos(n+1)\theta+2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin(n+1)\theta}{2\sin^2\frac{\theta}{2}}$

fortsettes

8

9 ii fortsat

$$= \frac{1}{2} + \frac{\sin(n+1)\frac{\theta}{2}\cos\frac{\theta}{2} - \cos(n+1)\theta\sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

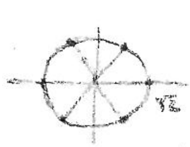
$$= \frac{1}{2} + \frac{\sin\frac{2n+1}{2}\theta}{2\sin\frac{\theta}{2}}, \quad \text{altså}$$

$$1 + \cos\theta + \cos 2\theta + \dots + \cos n\theta = \frac{1}{2} + \frac{\sin\frac{2n+1}{2}\theta}{2\sin\frac{\theta}{2}}$$

10

3 a $(-1)^{\frac{1}{3}} = \exp\frac{\pi + k2\pi}{3}i = \begin{cases} e^{i\frac{\pi}{3}} \\ e^{\pi i} \\ e^{-i\frac{\pi}{3}} \end{cases} = \begin{cases} \frac{1}{2}(1+i\sqrt{3}) \\ -1 \\ \frac{1}{2}(1-i\sqrt{3}) \end{cases}$

b $8^{\frac{1}{6}} = (2^3)^{\frac{1}{6}} = \sqrt{2}\exp i\frac{k2\pi}{6} = \begin{cases} \pm\sqrt{2} \\ \pm\frac{\sqrt{2}}{2}(1+i\sqrt{3}) \\ \pm\frac{\sqrt{2}}{2}(1-i\sqrt{3}) \end{cases}$



7 $c = \sqrt[n]{1}$, $c \neq 1$

$1 + c + c^2 + \dots + c^{n-1} = \frac{1-c^n}{1-c} = \frac{1-1}{1-c} = 0$

11

1 a $|z-2+i| \leq 1 \Leftrightarrow |x-2+i(y+1)| \leq 1$  nej

b $|2z+3| > 4 \Leftrightarrow |x+\frac{3}{2}+iy| > 2$  ja

c $\operatorname{Im} z > 1 \Leftrightarrow y > 1$  ja

d $\operatorname{Im} z = 1 \Leftrightarrow y = 1$  nej

e $0 \leq \arg z \leq \frac{\pi}{4} \wedge z \neq 0$  nej

f $|z-4| \geq |z|$  nej

2 a (hv. åben el. lukket)

3 a (begge)

7 a $z_n = i^n$, ingen fortætningspunkt

b 0

c $0 \leq \arg z < \frac{\pi}{2}$ $(z \neq 0)$, $0 \leq \arg z \leq \frac{\pi}{2}$

d $z_n = (-1)^n(1+i)\frac{n-1}{n}$, $\pm(1+i)$

12

1 a $f(z)=\frac{1}{z^2+1}$, $z\neq\pm i$

b $f(z)=\operatorname{Arg}(\frac{1}{z})=-\theta$, $z\neq 0$, $-\pi<-\theta\le\pi \Rightarrow -\pi\le\theta<\pi$

c $f(z)=\frac{z}{z+\bar z}$, $\operatorname{Re} z\neq 0$

d $f(z)=\frac{1}{1-|z|^2}$, $|z|\neq 1$

4 $f(z)=z+\frac{1}{z}=r\cos\theta+ir\sin\theta+\frac{1}{r}\cos\theta-i\frac{1}{r}\sin\theta$

$=(r+\frac{1}{r})\cos\theta+i(r-\frac{1}{r})\sin\theta$

14

1 $w=z^2$

$x^2-y^2=1$ $y=\frac{1}{2x}$ $u=x^2-y^2$

$x^2-y^2=2$ $y=\frac{1}{x}$ $v=2xy$

4 $w=\exp z=\rho e^{i\varphi}$, $ay=x$ $(a\neq 0)$

$\rho=e^x=e^{ay}$, $\varphi=y \Rightarrow \rho=e^{a\varphi}$ (log-spiral)

2 $w=\exp z$

15

3 a $\lim_{z\to z_0}\frac{1}{z^n}=\frac{1}{z_0^n}$, $z_0\neq 0$ (regneregler)

b $\lim_{z\to i}\frac{iz^3-1}{z+i}=\frac{i(-i)-1}{i+i}=\frac{1-1}{2i}=0$

c $\lim_{z\to z_0}\frac{P(z)}{Q(z)}=\frac{P(z_0)}{Q(z_0)}$, $Q(z_0)\neq 0$ (regneregler)

10 a $\lim_{z\to 0}\frac{4(\frac{1}{z})^2}{(\frac{1}{z}-1)^2}=\lim_{z\to 0}\frac{4}{(1-z)^2}=4 \Rightarrow \lim_{z\to\infty}\frac{4}{(z-1)^2}=4$

b $\lim_{z\to 1}\frac{(z-1)^3}{1}=0 \Rightarrow \lim_{z\to 1}\frac{1}{(z-1)^3}=\infty$

c $\lim_{z\to 0}\frac{\frac{1}{z}-1}{(\frac{1}{z})^2+1}=\lim_{z\to 0}\frac{z-z^2}{1+z^2}=0 \Rightarrow \lim_{z\to\infty}\frac{z^2+1}{z-1}=\infty$

20

3 $f(z)=\frac{1}{z}$; $\frac{\frac{1}{z+\Delta z}-\frac{1}{z}}{\Delta z}=\frac{z-z-\Delta z}{\Delta z\, z(z+\Delta z)}=-\frac{1}{z(z+\Delta z)}\to-\frac{1}{z^2}$ for $\Delta z\to 0$

altså $f'(z)=-\frac{1}{z^2}$

<u>20</u>

8 a $f(z) = \operatorname{Re} z = x$, $\dfrac{x + \Delta x - x}{\Delta z} = \dfrac{\Delta x}{\Delta z} \to \begin{cases} 1 & \text{længs x-aksen} \\ 0 & \text{– y-aksen} \end{cases}$

f ikke diff.

b $f(z) = \operatorname{Im} z = y$, $\dfrac{y + \Delta y - y}{\Delta z} = \dfrac{\Delta y}{\Delta z} \to \begin{cases} 0 & \text{længs x-aksen} \\ -1 & \text{længs y-aksen} \end{cases}$

f ikke diff.

<u>23</u>

3 a $f(z) = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$; $D = \mathbb{C}\setminus\{0\}$; $f \in C^1(D)$

C-R ok , $f'(z) = \dfrac{(x^2+y^2)1 - x\,2x}{(x^2+y^2)^2} + i\dfrac{y\,2x}{(x^2+y^2)^2} = -\dfrac{x^2-y^2-i2xy}{(x^2+y^2)^2}$

$= -\dfrac{\bar{z}^2}{|z|^4} = -\dfrac{\bar{z}^2}{z^2\bar{z}^2} = -\dfrac{1}{z^2}$

b $f(z) = x^2 + iy^2$, $D = \mathbb{C}$, $f \in C^1(D)$

C-R $2x = 2y \wedge 0 = -0$ kun opfyldt for $y = x$

$f'(x+ix) = 2x + i0\big|_{y=x} = 2x$

c $f(z) = z \operatorname{Im} z = xy + iy^2$, $D = \mathbb{C}$ $f \in C^1(D)$

C-R: $y = 2y \wedge x = 0$ kun opfyldt i $z = 0$

$f'(0) = y + i0\big|_{z=0} = 0$

4 a $f(z) = \frac{1}{z^4} = \frac{1}{r^4}e^{i(-4\theta)} = \frac{1}{r^4}\cos 4\theta - i\frac{1}{r^4}\sin 4\theta$, $D = \mathbb{C}\setminus\{0\}$, $f \in C^1(D)$

C-R: $r\left(-\frac{4}{r^5}\right)\cos 4\theta = -4\frac{1}{r^4}\cos 4\theta$ ok

$-4\frac{1}{r^4}\sin 4\theta = -r\left(+4\frac{1}{r^5}\sin 4\theta\right)$ ok

$f'(z) = e^{-i\theta}\left(-\frac{4}{r^5}\cos 4\theta + i\left(+4\frac{1}{r^5}\sin 4\theta\right)\right) = -4\frac{1}{r^5}e^{-i\theta}e^{-4i\theta} = -\frac{4}{z^5}$

b $f(z) = \sqrt{r}\,e^{i\frac{\theta}{2}}$, $D = r > 0 \wedge \alpha < \theta < \alpha + 2\pi$, $f \in C^1(D)$

C-R: $r\,\frac{1}{2}r^{-\frac{1}{2}}\cos\frac{\theta}{2} = \frac{1}{2}r^{\frac{1}{2}}\cos\frac{\theta}{2}$ ok

$-\frac{1}{2}r^{\frac{1}{2}}\sin\frac{\theta}{2} = -r\,\frac{1}{2}r^{-\frac{1}{2}}\sin\frac{\theta}{2}$ ok

$f'(z) = e^{-i\theta}\left(\frac{1}{2}r^{-\frac{1}{2}}\cos\frac{\theta}{2} + i\frac{1}{2}r^{-\frac{1}{2}}\sin\frac{\theta}{2}\right) = \frac{1}{2}r^{-\frac{1}{2}}e^{-i\theta}e^{i\frac{\theta}{2}} = \dfrac{1}{2f(z)}$

*

9 a $f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{r}v_\theta + i\left(-\frac{1}{r}u_\theta\right)\right) = -\frac{i}{re^{i\theta}}(u_\theta + iv_\theta) = -\frac{i}{z}(u_\theta + iv_\theta)$

b $f(z) = \frac{1}{z} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}\cos\theta - i\frac{1}{r}\sin\theta$, $z \neq 0$

$f'(z) = -\frac{i}{z}\left(-\frac{1}{r}\sin\theta - i\frac{1}{r}\cos\theta\right) = -\frac{1}{z}\frac{1}{r}(\cos\theta - i\sin\theta) = -\frac{1}{z}\frac{1}{re^{i\theta}}$

$= -\frac{1}{z}\frac{1}{z} = -\frac{1}{z^2}$

* 4 c $f(z) = e^{-\theta}\cos(\ln r) + i\,e^{-\theta}\sin(\ln r)$

$r\left(-\frac{1}{r}e^{-\theta}\sin(\ln r)\right) = -e^{-\theta}\sin(\ln r)$, $-e^{-\theta}\cos(\ln r) = -r\frac{1}{r}e^{-\theta}\cos(\ln r)$ C-R opfyldt

$f'(z) = e^{-i\theta}\left(-\frac{1}{r}e^{-\theta}\sin(\ln r) + i\frac{1}{r}e^{-\theta}\cos(\ln r)\right) = \dfrac{i\left(e^{-\theta}\cos(\ln r) + i\,e^{-\theta}\sin(\ln r)\right)}{re^{i\theta}} = \dfrac{i f(z)}{z}$

25

1 a $f(z) = 3x + y + i(3y - x)$

C-R: $3 = 3 \wedge 1 = -(-1)$ ok $\Rightarrow$ f hel

b $f(z) = \sin x \cosh y + i \cos x \sinh y$

C-R: $\cos x \cosh y = \cos x \cosh y \wedge \sin x \sinh y = -(-\sin x)\sinh y$ ok $\Rightarrow$ f hel

c $f(z) = e^{-y}\sin x - i e^{-y}\cos x$

C-R: $e^{-y}\cos x = -(-e^{-y})\cos x \wedge -e^{-y}\sin x = -(-e^{-y}(-\sin x))$ ok $\Rightarrow$ f hel

d $f(z) = (z^2-2)e^{-x}e^{-iy} = (z^2-2)g(z)$, z^2-2 hel fkt.

$g(z) = e^{-x}\cos y - i e^{-x}\sin y$

C-R: $-e^{-x}\cos y = -e^{-x}\cos y \wedge e^{-x}(-\sin y) = -(-(-e^{-x})\sin y)$ ok $\Rightarrow$ g hel

z^2-2 hel $\wedge$ $g(z)$ hel $\Rightarrow$ $f(z)$ hel

4 a $f(z) = \dfrac{2z+1}{z(z^2+1)}$, f analytisk i $\mathbb{C}\setminus\{0, \pm i\}$

b $f(z) = \dfrac{z^3+i}{z^2-3z+2}$, $z^2-3z+2=0$, $d = 9-8 = 1$, $z = \dfrac{3\pm1}{2} = \begin{cases}2\\1\end{cases}$

f analytisk i $\mathbb{C}\setminus\{1,2\}$

c $f(z) = \dfrac{z^2+1}{(z+2)(z^2+2z+2)}$, $z^2+2z+2=0$, $d = 4-8 = -4$, $z = \dfrac{-2\pm2i}{2} = -1\pm i$

f analytisk i $\mathbb{C}\setminus\{-2, -1\pm i\}$

6 i $g(z) = \ln r + i\theta$, $r>0 \wedge 0<\theta<2\pi$

C-R: $r\frac{1}{r} = 1 \wedge 0 = -0$ ok $\Rightarrow$ g analytisk i D

$g'(z) = e^{-i\theta}(\frac{1}{r} + i0) = \frac{1}{re^{i\theta}} = \frac{1}{z}$

ii $G(z) = g(z^2+1)$, $x, y > 0$,

Bemærk $\mathrm{Im}(z^2+1) = 2xy > 0$ i første kvadrant

$f(z) = z^2+1$ hel $\wedge$ $g(z)$ analytisk i 

$\Rightarrow$ $G(z) = g\circ f(z)$ analytisk i første kvadrant

$G'(z) = g'\circ f(z)\, f'(z) = \dfrac{1}{z^2+1}\, 2z = \dfrac{2z}{z^2+1}$

7 f analytisk i D, $f(z) = u(x,y) + i v(x,y) = u(x,y)$ (dvs. $v(x,y) = 0$)

C-R: $u_x = 0 \wedge u_y = -0 \Rightarrow u(x,y) = g(y) \wedge u(x,y) = h(x)$

$\Rightarrow u(x,y) =$ konst $\Rightarrow f(z) =$ konst.

1 a $u(x,y) = 2x(1-y)$ $\quad u_x = 2(1-y) \quad u_{xx} = 0$; $\quad u_y = -2x \quad u_{yy} = 0$ $\Big\}$ $\Delta u = 0$

$v_y = 2(1-y) \quad v(x,y) = 2y - y^2 + f(x)$
$v_x = -(-2x) \quad v(x,y) = x^2 + g(y)$ $\Big\}$ $v(x,y) = x^2 - y^2 + 2y + c$

b $u(x,y) = 2x - x^3 + 3xy^2$ $\quad u_x = 2 - 3x^2 + 3y^2 \quad u_{xx} = -6x$; $\quad u_y = 6xy \quad u_{yy} = 6x$ $\Big\}$ $\Delta u = 0$

$v_y = 2 - 3x^2 + 3y^2 \quad v(x,y) = 2y - 3x^2y + y^3 + f(x)$
$v_x = -6xy \quad v(x,y) = -3x^2y + h(y)$ $\Big\}$ $v(x,y) = 2y - 3x^2y + y^3 + c$

c $u(x,y) = \sinh x \sin y$ $\quad u_{xx} = u$; $\quad u_{yy} = -u$ $\Big\}$ $\Delta u = 0$

$v_y = \cosh x \sin y \quad v(x,y) = -\cosh x \cos y + h(x)$
$v_x = -\sinh x \cos y \quad v(x,y) = -\cosh x \cos y + g(y)$ $\Big\}$ $v(x,y) = -\cosh x \cos y + c$

d $u(x,y) = \dfrac{y}{x^2+y^2} \quad u_x = -\dfrac{2xy}{(x^2+y^2)^2} \quad u_y = \dfrac{x^2+y^2-y\,2y}{(x^2+y^2)^2} = \dfrac{x^2-y^2}{(x^2+y^2)^2}$

$$u_{xx} = -\frac{(x^2+y^2)^2 2y - 2xy\,2(x^2+y^2)2x}{(x^2+y^2)^4} = -\frac{2x^2y + 2y^3 - 8x^2y}{(x^2+y^2)^3} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$
$$u_{yy} = \frac{(x^2+y^2)^2(-2y) - (x^2-y^2)2(x^2+y^2)2y}{(x^2+y^2)^4} = \frac{-2x^2y - 2y^3 - 4x^2y + 4y^3}{(x^2+y^2)^3} = \frac{-6x^2y + 2y^3}{(x^2+y^2)^3}$$
$\Big\}$ $\Delta u = 0$

$v_y = -\dfrac{2xy}{(x^2+y^2)^2} \quad v(x,y) = \dfrac{x}{x^2+y^2} + g(x)$
$v_x = -\dfrac{x^2-y^2}{(x^2+y^2)^2} \quad v(x,y) = \dfrac{x}{x^2+y^2} + h(y)$ $\Big\}$ $v(x,y) = \dfrac{x}{x^2+y^2} + c$

3 v harm. konj til u $\quad u_x = v_y \wedge u_y = -v_x$
u — — — v $\quad v_x = u_y \wedge v_y = -u_x$

$u_x = v_y \wedge u_x = -v_y \Rightarrow u_x = 0 \Rightarrow u(x,y) = g(y)$
$u_y = v_x \wedge u_y = -v_x \Rightarrow u_y = 0 \Rightarrow u(x,y) = h(x)$ $\Big\}$ $u(x,y) = c_1$
analogt $v(x,y) = c_2$ $\Big\}$ $f(z) = c$

4 $f(z)$ analytisk i $D \Leftrightarrow -if(z)$ analytisk i D (regneregler)
$f(z) = u + iv$ analytisk $\Leftrightarrow v$ er harm. konj til u
$-if(z) = v - iu$ analytisk $\Leftrightarrow -u$ er harm. konj. til v

6 $u(r,\theta) = \ln r, \quad r > 0, \quad 0 < \theta < 2\pi$

i $u_r = \frac{1}{r} \quad u_{rr} = -\frac{1}{r^2}$; $\quad u_\theta = 0 \quad u_{\theta\theta} = 0$ $\Big\}$ $r^2 u_{rr} + r u_r + u_{\theta\theta} = -1 + 1 + 0 = 0$ $\quad u$ harm.

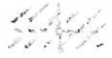
ii $v_\theta = 1 \quad v = \theta + g(r)$
$v_r = -\frac{1}{r} 0 \quad v = h(\theta)$ $\Big\}$ $v(r,\theta) = \theta + c$

2 $f_1(z) = \sqrt{r}\, e^{i\frac{\theta}{2}}$, $r>0$, $0<\theta<\pi$ 

$f_2(z) = \sqrt{r}\, e^{i\frac{\theta}{2}}$, $r>0$, $\frac{\pi}{2}<\theta<2\pi$  $\Big\}$ f_2 er analytisk fortsættelse af f_1.

$f_3(z) = \sqrt{r}\, e^{i\frac{\theta}{2}}$, $r>0$, $\pi<\theta<\frac{5\pi}{2}$  $\Big\}$ f_3 er analytisk fortsættelse af f_2

Bemærk, at $f_3(z) = -f_1(z)$ i første kvadrant

3 $f_4(z) = \sqrt{r}\, e^{i\frac{\theta}{2}}$, $r>0$, $-\pi<\theta<\pi$  f_4 er analytisk fortsættelse af f_1 over R_+

da f_1 og f_4 er analytiske og f_1 og f_4 stemmer overens i første kvadrant

4 $f(z) = e^x e^{-iy}$ $D = \mathbb{C}$ (spejl om x-aksen)

$f(x)$ reel på x-aksen ($f(x) = e^x$) $\Big\}$ $f(\bar{z}) = \overline{f(z)}$

29

1 a $\exp(2 \pm 3\pi i) = e^2(-1 + i0) = -e^2$

b $\exp\left(\frac{2+\pi i}{4}\right) = e^{\frac{1}{2}} \frac{\sqrt{2}}{2}(1+i) = \sqrt{\frac{e}{2}}(1+i)$

c $\exp(z + \pi i) = \exp z \,(-1) = -\exp z$

4 $f(z) = \exp z^2$

i z^2 hol $\wedge$ exp hol $\Rightarrow$ $\exp z^2$ hol

$f'(z) = \exp z^2 \,(2z) = 2z \exp z^2$

ii $\exp z^2 = e^{x^2-y^2}\cos 2xy + i\, e^{x^2-y^2}\sin 2xy = u + iv$

$u_x = 2x e^{x^2-y^2}\cos 2xy - 2y e^{x^2-y^2}\sin 2xy$

$v_y = -2y e^{x^2-y^2}\sin 2xy + 2x e^{x^2-y^2}\cos 2xy$ $\Big\}\; u_x = v_y$

$u_y = -2y e^{x^2-y^2}\cos 2xy - 2x e^{x^2-y^2}\sin 2xy$

$v_x = 2x e^{x^2-y^2}\sin 2xy + 2y e^{x^2-y^2}\cos 2xy$ $\Big\}\; u_y = -v_x$

$\Big\}$ C-R opfyldt overalt

$\Rightarrow$ f hol

$f'(z) = u_x + i v_x$

$= 2x e^{x^2-y^2}\cos 2xy - 2y \sin 2xy + i(2x e^{x^2-y^2}\sin 2xy + 2y\cos 2xy)$

$= 2(x+iy)(e^{x^2-y^2}\cos 2xy + i\, e^{x^2-y^2}\sin 2xy)$

$= 2z \exp z^2$

9 $\overline{\exp(iz)} = \overline{\exp(-y+ix)} = e^{-y}\,\overline{(\cos x + i\sin x)} = e^{-y}(\cos x - i\sin x)$

$\exp(i\bar{z}) = \exp(i(x-iy)) = \exp(y+ix) = e^{y}(\cos x + i\sin x)$

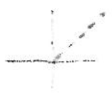
$\overline{\exp(iz)} = \exp(i\bar{z}) \Rightarrow y = 0 \wedge \sin x = 0 \Leftrightarrow x = n\pi,\ n \in \mathbb{Z} \wedge y = 0$

31

2 a $\log e = \ln e + i\, 2n\pi = 1 + n2\pi i \quad , n \in \mathbb{Z}$

b $\log i = \ln 1 + i\left(\frac{\pi}{2} + n2\pi\right) = \left(\frac{\pi}{2} + n2\pi\right) \quad , n \in \mathbb{Z}$

c $\log(-1 + i\sqrt{3}) = \ln 2 + i\left(\frac{2\pi}{3} + n2\pi\right) \quad , n \in \mathbb{Z}$

4 a 

$\log(i^2) = \log(-1) = \pi i$

$2\log i = 2 i \frac{\pi}{2} = \pi i$

b 

$\log(i^2) = \log(-1) = \pi i$

$2\log i = 2 i \frac{5\pi}{2} = 5\pi i$

<u>32</u>

3 a $\log\frac{z_1}{z_2} = \ln\left|\frac{z_1}{z_2}\right| + i\arg\frac{z_1}{z_2} = \ln|z_1| - \ln|z_2| + i(\arg z_1 - \arg z_2)$

$= \ln|z_1| + i\arg z_1 - (\ln|z_2| + i\arg z_2) = \log z_1 - \log z_2$

b $\log\left(\frac{1}{z}\right) = \ln\frac{1}{|z|} + i\arg\frac{1}{z} = -\ln|z| - i\arg z = -\log z$

$\log\left(\frac{z_1}{z_2}\right) = \log\left(z_1\frac{1}{z_2}\right) = \log z_1 + \log\frac{1}{z_2} = \log z_1 - \log z_2$

5 $z^{\frac{1}{n}} = \exp\left(\frac{1}{n}\log z\right)$, $n \in \mathbb{Z}_+$, sæt $m = -n$

$z^{\frac{1}{n}} = \left(z^{\frac{1}{m}}\right)^{-1} = \frac{1}{z^{\frac{1}{m}}} = \frac{1}{\exp\left(\frac{1}{m}\log z\right)} = \exp\left(-\frac{1}{m}\log z\right) = \exp\left(\frac{1}{n}\log z\right)$

<u>33</u>

2 a p.v. $i^i = \exp(i\operatorname{Log} i) = \exp\left(i\cdot i\frac{\pi}{2}\right) = \exp\left(-\frac{\pi}{2}\right)$

b p.v. $\left(\frac{e}{2}(-1-i\sqrt{3})\right)^{3\pi i} = \exp\left(3\pi i\operatorname{Log}\left(\frac{e}{2}(-1-i\sqrt{3})\right)\right)$

$= \exp\left(3\pi i\left(\ln\frac{e\cdot 2}{2} + i\left(0 - \frac{2\pi}{3}\right)\right)\right) = \exp(3\pi i + 2\pi^2) = -\exp(2\pi^2)$

c p.v. $(1-i)^{4i} = \exp(4i\operatorname{Log}(1-i)) = \exp\left(4i\left(\ln\sqrt{2} + i\left(-\frac{\pi}{4}\right)\right)\right)$

$= \exp(\pi + i\,2\ln 2) = e^{\pi}(\cos 2\ln 2 + i\sin 2\ln 2)$

5 p.v. $z_0^{\frac{1}{n}} = \exp\left(\frac{1}{n}\operatorname{Log} z_0\right) = \exp\left(\frac{1}{n}\ln|z_0| + i\operatorname{Arg} z_0\right)$

$= \sqrt[n]{|z_0|}\exp\left(i\frac{\operatorname{Arg} z_0}{n}\right)$

<u>34</u>

4 a $\cos(z_1 + z_2) = \cos z_1\cos z_2 - \sin z_1\sin z_2$

$z_1 = z \wedge z_2 = -z \Rightarrow 1 = \cos^2 z - \sin z(-\sin z)$

$\Leftrightarrow \cos^2 z + \sin^2 z = 1$

b $f(z) = \cos^2 z + \sin^2 z - 1$, f hel

$f(x) = 0 \Rightarrow f(z) = 0$ i $\mathbb{C}$ (lemma i afsn. 27)

$\Leftrightarrow \cos^2 z + \sin^2 z = 1$

9 $|\sin z|^2 = \sin^2 x + \sinh^2 y = -\cos^2 x + \cosh^2 y$ ($-1+1$ og omskrivning)

$|\cos z|^2 = \cos^2 x + \sinh^2 y = -\sin^2 x + \cosh^2 y$ (do.)

a $|\sinh y| \leq |\sin z| \leq \cosh y$ (cosh er lige)

b $|\sinh y| \leq |\cos z| \leq \cosh y$ (do.)

34

11 i $\sin\bar z = \sin(x-iy) = \sin x\cosh y - i\cos x\sinh y$ (jævnfør ovenfor)

$u_x = \cos x\cosh y$, $v_y = -\cos x\cosh y$
$u_y = \sin x\sinh y$, $v_x = \sin x\cosh y$ } C-R ikke opfyldt i C

$\Rightarrow$ $\sin\bar z$ ikke analytisk i nogen punkt

ii $\cos\bar z = \cos(x-iy) = \cos x\cosh y + i\sin x\sinh y$

$u_x = -\sin x\cosh y$, $v_y = \sin x\cosh y$
$u_y = \cos x\sinh y$, $v_x = \cos x\sinh y$ } C-R ikke opfyldt i C

$\Rightarrow$ $\cos\bar z$ ikke analytisk i nogen punkt

35

6 a $|\cosh z|^2 = \sinh^2 x + \cos^2 y = \cosh^2 x - \sin^2 y$ jf. (12) afsn. 35 samt omskrivning

$|\sinh x| \le |\cosh z| \le \cosh x$

b $|\sinh y| \le |\cos z| \le \cosh y$, jf. opg. 34 9 b

$z := iz \Rightarrow |\sinh x| \le |\cos iz| \le \cosh x$

$\iff |\sinh x| \le |\cosh z| \le \cosh x$, jf. (4) afsn. 35

13 a $f(z) = \cosh^2 z - \sinh^2 z - 1$ f hel

$f(x) = \cosh^2 x - \sinh^2 x - 1 = 0 \Rightarrow f(z) = 0$ i C, jf. Lemma i afsn. 27

$\Rightarrow \cosh^2 z - \sinh^2 z = 1$

b $f(z) = \sinh z + \cosh z - e^z$, f hel

$f(x) = \sinh x + \cosh x - e^x = 0 \Rightarrow f(z) = 0$ i C, do.

$\Rightarrow \sinh z + \cosh z = e^z$ i C

36

2 a $\sin z = 2 \iff \sin x\cosh y + i\cos x\sinh y = 2 + i0$

$y = 0 \Rightarrow \sin x = 2 \Rightarrow x = \arcsin 2$ umulig

$x = \frac{\pi}{2} + n2\pi \Rightarrow \cosh y = 2 \Rightarrow \frac{e^y + e^{-y}}{2} = 2 \iff e^{2y} - 4e^y + 1 = 0$

$\iff e^y = \frac{4 \pm 2\sqrt3}{2} = 2 \pm \sqrt3 \Rightarrow y = \ln(2\pm\sqrt3) = \pm\ln(2+\sqrt3)$

$z = \frac{\pi}{2} + n2\pi \pm i\ln(2+\sqrt3)$, $n \in \mathbb{Z}$

b $\sin z = 2 \iff z = \arcsin 2 = -i\log(i2 + (1-4)^{\frac12})$

$= -i\log((2\pm\sqrt3)i) = -i(\ln(2\pm\sqrt3) + i(\frac{\pi}{2} + n2\pi))$

$= \frac{\pi}{2} + n2\pi \pm i\ln(2+\sqrt3)$, $n \in \mathbb{Z}$

36

5 $w = \tan z \iff z = \arctan w$

$$w = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} \iff iw = \frac{e^{2iz} - 1}{e^{2iz} + 1}$$

$$\Rightarrow (1 - iw) e^{2iz} = 1 + iw \Rightarrow e^{2iz} = \frac{1 + iw}{1 - iw} = \frac{i - w}{i + w}$$

$$\Rightarrow 2iz = \log \frac{i - w}{i + w} \Rightarrow z = \frac{1}{2i} \log \frac{i-w}{i+w} = \frac{i}{2} \log \frac{i+w}{i-w}$$

$w \leftrightarrow z$: $\arctan z = \frac{i}{2} \log \frac{i+z}{i-z}$

38

1 a $\frac{d}{dt} w(-t) = \frac{d}{dt}(u(-t) + i v(-t)) = -u'(-t) - i v'(-t) = -w'(-t)$

b $\frac{d}{dt}(w(t))^2 = \frac{d}{dt}(u(t) + i v(t))^2 = \frac{d}{dt}\left((u(t))^2 - (v(t))^2 + i\, 2u(t)v(t)\right)$

$$= 2u(t)u'(t) - 2v(t)v'(t) + i\,2\left(u'(t)v(t) + u(t)v'(t)\right)$$

$$= 2\left(u(t) + iv(t)\right)\left(u'(t) + iv'(t)\right) = 2\, w(t)\, w'(t)$$

2 a $\int_1^2 \left(\frac{1}{t} - i\right)^2 dt = \int_1^2 \left(\frac{1}{t^2} - 1 - \frac{2i}{t}\right) dt = \left[-\frac{1}{t} - t - 2i \ln t\right]_1^2$

$$= -\frac{1}{2} - 2 - 2i \ln 2 + 1 + 1 + 0 = -\frac{1}{2} - i \ln 4$$

b $\int_0^{\pi/6} e^{i2t}\, dt = \left[\frac{1}{2i} e^{i2t}\right]_0^{\pi/6} = \frac{1}{2i}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2} - 1\right) = \frac{\sqrt{3}}{4} + \frac{i}{4}$

c $\int_0^\infty e^{-zt}\, dt = \left[-\frac{1}{z} e^{-zt}\right]_0^\infty = \frac{1}{z} - 0 = \frac{1}{z}$

3 $m \neq n$: $\int_0^{2\pi} e^{im\theta} e^{-in\theta}\, d\theta = \int_0^{2\pi} e^{i(m-n)\theta}\, d\theta = \left[\frac{1}{i(m-n)} e^{i(m-n)\theta}\right]_0^{2\pi}$

$$= \frac{1}{i(m-n)}(1 - 1) = 0$$

$m = n$: $\int_0^{2\pi} e^{in\theta} e^{-in\theta} = \int_0^{2\pi} d\theta = 2\pi$

39

2 c 

$z = 2e^{i\theta}, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$z = \sqrt{4 - y^2} + iy, \quad -2 \le y \le 2 \quad (x^2 + y^2 = 2^2)$

$$\theta = \arctan \frac{y}{x} = \arctan \frac{y}{\sqrt{4 - y^2}} = \varphi(y)$$

$$\varphi'(y) = \frac{1}{1 + \frac{y^2}{4 - y^2}} \cdot \frac{\sqrt{4-y^2} \cdot 1 - y \frac{1}{2\sqrt{4-y^2}}(-2y)}{4 - y^2} = \frac{4 + 2y^2}{4\sqrt{4 - y^2}} > 0 \text{ for } -2 < y < 2$$

<u>39</u>

5 $f(z) = u(x,y) + i v(x,y)$ analytisk i D, $z = z(t) = x(t) + i y(t)$

$f \circ z(t) = u(x(t), y(t)) + i(v(x(t), y(t)))$

$(f \circ z)' = u_x x' + u_y y' + i(v_x x' + v_y y')$

$= u_x x' - v_x y' + i(v_x x' + u_x y')$, C-R anvendt

$= (u_x + i v_x)(x' + i y')$

$= (f' \circ z) z'$

6 $y(x) = \begin{cases} x^3 \sin\frac{\pi}{x}, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$

a $z = x + i y(x)$, $0 \le x \le 1$

$x > 0$: $y = 0 \iff x^3 \sin\frac{\pi}{x} = 0 \iff \sin\frac{\pi}{x} = 0$

$\iff \frac{\pi}{x} = n\pi \iff x = \frac{1}{n}$, $n \in \mathbb{Z}_+$

b i $\left| x^3 \sin\frac{\pi}{x} - 0 \right| \le x^3 < \varepsilon$ for $0 < x < \sqrt[3]{\varepsilon}$

$\Rightarrow y(x)$ kont i 0

ii $y'(x) = 3x^2 \sin\frac{\pi}{x} + x^3 \cos\frac{\pi}{x}\left(-\frac{\pi}{x^2}\right)$

$= 3x^2 \sin\frac{\pi}{x} - \pi x \cos\frac{\pi}{x}$, $0 < x \le 1$

$\left| 3x^2 \sin\frac{\pi}{x} - \pi x \cos\frac{\pi}{x} - 0 \right| \le 3x^2 + \pi x < \varepsilon$

for $3x^2 + \pi x - \varepsilon < 0 \iff x < \frac{-\pi + \sqrt{\pi + 12\varepsilon}}{6}$

$\Rightarrow y'(x)$ kont i 0, $y'(0) = 0$

$\Rightarrow$ $C: x + i y(x)$ er diff. kurve (glatt kurve)

<u>42</u>

1 a 

$z = 2e^{i\theta}$, $0 \le \theta \le \pi$,

$\int_C \frac{z+2}{z}\,dz = \int_0^\pi \frac{2e^{i\theta}+2}{2e^{i\theta}}\, 2ie^{i\theta}\,d\theta = \int_0^\pi 2i(e^{i\theta}+1)\,d\theta$

$= 2\left[e^{i\theta} + i\theta\right]_0^\pi = 2(-1 + i\pi - 1 - 0) = -4 + 2\pi i$

Fortsetter

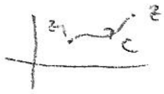
<u>42</u>

1 b  $z = 2e^{i\theta}, \ \pi \le \theta \le 2\pi$

$$\int_C \frac{z+2}{z}\,dz = 2\left[e^{i\theta} + i\theta\right]_{\pi}^{2\pi} = 2\left(1 + i2\pi - (-1) - i\pi\right)$$
$$= 4 + 2\pi i$$

c  $z = 2e^{i\theta}, \ 0 \le \theta \le 2\pi$

$$\int_C \frac{z+2}{z}\,dz = 2\left[e^{i\theta} + i\theta\right]_{0}^{2\pi} = 2\left(1 + i2\pi - 1 - 0\right)$$
$$= 4\pi i$$

5  C = vilk. kurve fra z_1 til z_2

$f(z) = 1$

$$\int_C f(z)\,dz = \int_{t_1}^{t_2} f\circ z(t)\, z'(t)\,dt = \int_{t_1}^{t_2} z'(t)\,dt = \left[z(t)\right]_{t_1}^{t_2} = z_2 - z_1$$

8 $C : |z| = 1, \ z = e^{i\theta}, \ 0 \le \theta \le 2\pi$

$$\int_C z^m \bar{z}^n\,dz = \int_0^{2\pi} e^{im\theta} e^{-in\theta}\, i e^{i\theta}\,d\theta = \begin{cases} 0 & \text{for } m \neq n-1 \\ 2\pi & \text{for } m = n-1 \end{cases} \quad \text{jf. opg } \underline{38}\ 3$$

10  $z = z_0 + Re^{i\theta}, \ -\pi \le \theta \le \pi$

 $z = Re^{i\theta}, \ -\pi \le \theta \le \pi$

a $$\int_{C_0} f(z - z_0)\,dz = \int_{-\pi}^{\pi} f(Re^{i\theta})\, iRe^{i\theta}\,d\theta = iR\int_{-\pi}^{\pi} f(Re^{i\theta})\,d\theta$$

$$\int_C f(z)\,dz = \int_{-\pi}^{\pi} f(Re^{i\theta})\, iRe^{i\theta}\,d\theta = iR\int_{-\pi}^{\pi} f(Re^{i\theta})\,d\theta$$

$$\Rightarrow \int_{C_0} f(z - z_0)\,dz = \int_C f(z)\,dz$$

b $$\int_{C_0} (z - z_0)^{n-1}\,dz = \int_C z^{n-1}\,dz = 0, \quad n \in \mathbb{Z}\setminus\{0\}, \quad \text{jf. (3) opg. 42}$$

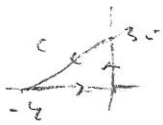
$$\int_{C_0} \frac{dz}{z - z_0} = \int_C \frac{dz}{z} = 2\pi i, \quad \text{jf. (6) opg. 42}$$

<u>43</u>

1  $$\left|\int_C \frac{dz}{z^2 - 1}\right| \le \int_C \frac{|dz|}{|z^2 - 1|} \le \frac{1}{3}\int_C |dz| \le \frac{1}{3}\, 2\,\frac{\pi}{2} = \frac{\pi}{3}$$

idet $|z^2 - 1| \ge \left||z^2| - 1\right| = 4 - 1 = 3$

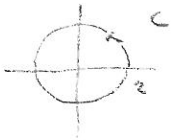
<u>43</u>

3 

$$\left|\int_C (e^z-\bar z)\,dz\right| \le \int_C |e^z-\bar z|\,dz \le \int_C (e^z+|z|)\,dz$$

$$\le \int_{-4}^{0}(e^x+|x|)\,dx + \int_0^3 (1+|y|)\,dy + \int_0^{-4}\left(e^x+\sqrt{x^2+(\tfrac{3}{4}x+3)^2}\right)dx$$

$$\le (1+4)4+(1+3)3+(1+4)5 = 20+12+25 = 57 < 60$$

5 

$$\left|\int_C \frac{\operatorname{Log} z}{z^2}\,dz\right| \le \int_C \left|\frac{\operatorname{Log} z}{z^2}\right| dz \le \int_C \frac{\ln R+\pi}{R^2}\,dz$$

$$\le \frac{\ln R+\pi}{R^2}\,2\pi R = 2\pi\,\frac{\pi+\ln R}{R^2} \longrightarrow 0 \quad \text{for } R\to\infty$$

<u>45</u>

2 a $$\int_i^{\frac{i}{2}} e^{\pi z}\,dz = \left[\frac{1}{\pi}e^{\pi z}\right]_i^{\frac{i}{2}} = \frac{1}{\pi}\left(e^{i\frac{\pi}{2}}-e^{i\pi}\right) = \frac{1}{\pi}(1+i)$$

b $$\int_0^{\pi+2i}\cos\left(\frac{z}{2}\right)dz = \left[2\sin\frac{z}{2}\right]_0^{\pi+2i} = 2\sin\left(\frac{\pi}{2}+i\right) = 2\cos i$$

$$= e+\frac{1}{e}$$

c $$\int_1^3 (z-2)^3\,dz = \left[\frac{(z-2)^4}{4}\right]_1^3 = \frac14-\frac14 = 0$$

3 

$$\int_{C_0}(z-z_0)^{n-1}\,dz = \left[\frac{(z-z_0)^n}{n}\right]_c^c = 0, \quad n\in\mathbb{Z}\setminus\{0\}$$

5 

Log z ikke def i $z=-1$  $-\pi < \arg z < \pi$

I stedet benyttes log z  $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$

stemmer overens m Log z

overalt på nær $z=-1$

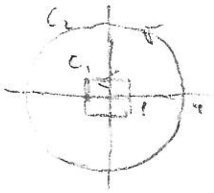
$$\int_{-1}^{1} z^i\,dz = \left[\frac{z^{i+1}}{i+1}\right]_{-1}^{1} = \frac{1}{i+1}\left[\exp\left((i+1)(\ln|z|+i\arg z)\right)\right]_{-1}^{1}$$

$$= \frac{1-i}{2}\left(\exp\left((1+i)(0+i0)\right)-\exp\left((1+i)(0+i\pi)\right)\right)$$

$$= \frac12(1-i)\left(1-(-1)e^{-\pi}\right)$$

$$= \frac12(1-i)\left(1+e^{-\pi}\right)$$

2



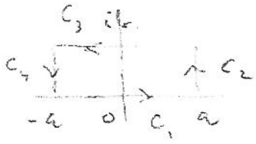
$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)$, når $f(z)$ analytisk i det lukkede område mellem C_1 og C_2

a $\quad f(z) = \frac{1}{3z^2+1}$; $3z^2+1 = 0 \Leftrightarrow z = \pm\frac{\sqrt{3}}{3}i$, $|z| < 1$

b $\quad f(z) = \frac{z+2}{\sin\frac{z}{2}}$; $\sin\frac{z}{2} = 0 \Leftrightarrow z = p2\pi$. $|z|<1 \vee |z|>4$

c $\quad f(z) = \frac{z}{1-e^z}$; $1-e^z = 0 \Leftrightarrow z = p2\pi i$, $|z| > 4$

4



$C = C_1 + C_2 + C_3 + C_4 \qquad f(z) = e^{-z^2}$, $-z^2 = y^2 - x^2 - i2xy$

$C_1: z = x$, $C_2: z = a+iy$. $C_3: z = x+ib$, $C_4: z = -a+iy$

a
$$\int_{C_1+C_2} e^{-z^2}dz = 2\int_0^a e^{-x^2}dx - 2\int_0^a e^{b^2-x^2-i2bx}dx$$
$$= 2\int_0^a e^{-x^2}dx - 2e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx + 0$$
$$\int_{C_2+C_4} e^{-z^2}dz = \int_0^b e^{y^2-a^2-i2ay}\,i\,dy - \int_0^b e^{y^2-a^2+i2ay}\,i\,dy$$
$$= ie^{-a^2}\int_0^b e^{y^2}e^{-i2ay}dy - ie^{-a^2}\int_0^b e^{y^2+i2ay}dy$$
$$= -2e^{a^2}\int_0^b e^{y^2}\sin 2by\,dy$$
$$\int_C e^{-z^2}dz = 2\left(\int_0^a e^{-x^2}dx - e^{b^2}\int_0^a e^{-x^2}\cos 2bx\,dx - e^{a^2}\int_0^b e^{y^2}\sin 2by\,dy\right.$$

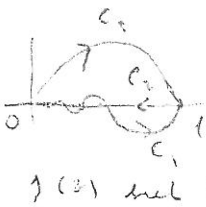
e^{-z^2} analytisk $\Rightarrow \int_C e^{-z^2}dz = 0 \Rightarrow$
$$\int_0^a e^{-x^2}\cos 2bx\,dx = e^{-b^2}\int_0^a e^{-x^2}dx - e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2by\,dy$$

b $\quad \int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ (velkendt)

$$\left|\int_0^b e^{y^2}\sin 2by\,dy\right| \le \int_0^b |e^{y^2}|\,|\sin 2by|\,dy \le \int_0^b e^{y^2}dy$$
$$\Rightarrow \left|e^{-(a^2+b^2)}\int_0^b e^{y^2}\sin 2by\,dy\right| \le e^{-a^2}e^{-b^2}\int_0^b e^{y^2}dy \to 0 \text{ for } a\to\infty$$

$$\Rightarrow \int_0^\infty e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}$$

<u>49</u>

5 

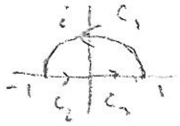
$C_1: \; z = x + iy(x)$, $y(x) = \begin{cases} x^3 \sin\frac{\pi}{x}, & 0 < x \le 1 \\ 0 & x = 0 \end{cases}$

$y(x)$ og $y'(x)$ kont. i 0, jf. opg 38 c

$f(z)$ hel fkt.

$C - C_1$: $\int_{C_1} f(z)dz = \int_{C_3} f(z)dz = -\int_{C_2} f(z)dz \Rightarrow \int_{C_1+C_2} f(z)dz = 0$

Bemærk C_2 skærer C_1 uendeligt mange gange

6 

$C_1: \; z = e^{i\theta}, \; 0 \le \theta \le \pi$, $C_2: \; z = x$, $C_3: \; z = x$

$f(z) = z^{\frac{1}{2}} = \sqrt{r}\, e^{i\frac{\theta}{2}}, \quad r > 0, \; -\frac{\pi}{2} < \theta < \frac{3\pi}{2}$

$$\int_{C_1} f(z)\,dz = \int_0^\pi e^{i\frac{\theta}{2}}\, i e^{i\theta}\, d\theta = i\int_0^\pi e^{i\frac{3\theta}{2}}\,d\theta = i\left[\frac{2}{3i} e^{i\frac{3\theta}{2}}\right]_0^\pi$$
$$= \frac{2}{3}(-i-1) = -\frac{2}{3}(1+i)$$

$$\int_{C_2} f(z)\,dz = \int_{-1}^0 \sqrt{x}\,dx = \left[\frac{2}{3}x\sqrt{x}\right]_{-1}^0 = \frac{2}{3}i$$

$$\int_{C_3} f(z)\,dz = \int_0^1 \sqrt{x}\,dx = \left[\frac{2}{3}x\sqrt{x}\right]_0^1 = \frac{2}{3}$$

$$\int_C f(z)\,dz = \int_{C_1+C_2+C_3} f(z)\,dz = -\frac{2}{3}(1+i) + \frac{2}{3}i + \frac{2}{3} = 0$$

($f(z)$ ikke analytisk i $z = 0$)

7 

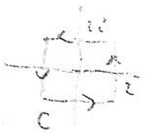
Afsn. 46 formel (4):

$$\int_C f(z)\,dz = \int_R (-v_x - u_y)\,dA + i\int_R (u_x - v_y)\,dA$$

$$\int_C \bar{z}\,dz = \int_R (-0-0)\,dA + i\int_R (1-(-1))\,dA = 2i\int_R dA = 2iA_R$$

$$\Rightarrow \quad A_R = \frac{1}{2i}\int \bar{z}\,dz$$

<u>52</u>

1 

a $\displaystyle\int_C \frac{e^{-z}}{z - i\frac{\pi}{2}}\,dz = 2\pi i\, e^{-i\frac{\pi}{2}} = 2\pi i(-i) = 2\pi$

b $\displaystyle\int_C \frac{\cos z}{z(z^2+8)}\,dz = 2\pi i\,\frac{1}{8} = i\frac{\pi}{4}$

c $\displaystyle\int_C \frac{z\,dz}{2z+1}\,dz = \frac{1}{2}2\pi i\left(-\frac{1}{2}\right) = -\frac{\pi i}{2}$ d $\displaystyle\int_C \frac{\cosh z}{z^4}\,dz = 2\pi i\cdot 0 = 0$

e $\displaystyle\int_C \frac{\tan\frac{z}{2}}{(z-x_0)^2}\,dz = 2\pi i\,\frac{1}{2}\,\frac{1}{\cos^2\frac{x_0}{2}} = i\frac{\pi}{\cos^2\frac{x_0}{2}} = i\pi\left(1+\tan^2\frac{x_0}{2}\right)$

52

2 

a $\int_C \frac{1}{z^2+4}\,dz = \int_C \frac{dz}{(z+2i)(z-2i)} = 2\pi i\,\frac{1}{4i} = \frac{\pi}{2}$

b $\int_C \frac{1}{(z^2+4)^2}\,dz = \int_C \frac{dz}{(z+2i)^2(z-2i)^2} = 2\pi i\,\frac{-2}{(4i)^3} = \frac{\pi}{16}$

4 

$g(s) = \frac{s^3+2s}{(s-z)^3}$ $\quad \frac{d}{ds}(s^3+2s) = 3s^2+2 \quad \frac{d^2}{ds^2} = 6s$

i z inden for C: $\int_C g(s)\,ds = 2\pi i\,6z = i\,6\pi z$

ii z uden for C: $\int_C g(s)\,ds = 0$, da $g(s)$ analytisk inden for C

5 

$\int_C \frac{f'(z)\,dz}{z-z_0} = 2\pi i\,f'(z_0)$, jf. afsn 50 formel (2) med $f := f'$

$\int_C \frac{f(z)\,dz}{(z-z_0)^2} = 2\pi i\,f'(z_0)$, jf. afsn 51 formel (5) med $z := z_0$
og $s := z$

*

7 

C: $z = e^{i\theta}$, $-\pi \le \theta \le \pi$ $\quad$ a reel konstant

$\int_C \frac{e^{az}}{z}\,dz = 2\pi i\,e^{a0} = 2\pi i$

$\int_C \frac{e^{az}}{z}\,dz = \int_{-\pi}^{\pi} \frac{e^{a(\cos\theta + i\sin\theta)}}{e^{i\theta}}\, i e^{i\theta}\,d\theta$

$= i\int_0^{2\pi} e^{a\cos\theta}\left(\cos(a\cos\theta) + i\sin(a\cos\theta)\right)d\theta$

$\Rightarrow \int_{-\pi}^{\pi} e^{a\cos\theta}\cos(a\cos\theta)\,d\theta = 2\pi$

$\Rightarrow \int_0^{\pi} e^{a\cos\theta}\cos(a\cos\theta)\,d\theta = \pi$ (cos lige)

54

1 $f(z) = u(x,y) + i v(x,y)$, f hel, $u(x,y) \le u_0$ for alle z

$g(z) = \exp f(z)$, $|g(z)| = |\exp u(x,y)| \le e^{u_0}$

$g(z)$ konst. jf. Liouvilles sætn

$\Rightarrow u(x,y)$ konst for alle (x,y)

*  $\cdot z_0$ begge integraler 0

2 $P(z) = a_0 + a_1 z + \dots + a_n z^n$

$w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \dots + \frac{a_{n-1}}{z}$, dvs $P(z) = (w + a_n) z^n$

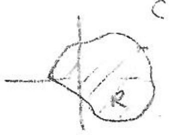
vælg R, så $\left|\frac{a_j}{z^{n-j}}\right| < \left|\frac{a_n}{n}\right|$ for alle j, $|z| > R$

$\Rightarrow |w| < |a_n| \Rightarrow |a_n + w| \le |a_n| + |a_n| = 2|a_n|$, $|z| > R$

$\Rightarrow |P(z)| \le 2|a_n| R^n$, $|z| > R$

Fra afs. 53 formel (5): $\frac{1}{2}|a_n| R^n < |P(z)|$ $\Bigg\} \Rightarrow$

$$\frac{1}{2}|a_n| R^n < |P(z)| < 2|a_n| R^n$$

4 

$0 \in R$

eks: $f(z) = z$, $f(z) = 0$ for $z = 0$

$\Rightarrow \min |f(z)| = 0$ for $z = 0$

dvs indenfor C

5 

$f(z) = (z+1)^2$

$|f(z)| = |z+1|^2 = |z-(-1)|^2$

$\max |f(z)| = 9$ for $z = 2$ $\quad \min |f(z)| = 1$ for $z = 0$

6 

$f(z) = u(x,y) + i\,v(x,y)$ ikke konst.

$f(z)$ analytisk i R

$g(z) = e^{-f(z)} \neq 0$ for alle z

$|g(z)| = \exp(-u(x,y))$ har maksimum på C

$\Rightarrow -u(x,y)$ har maksimum på C

$\Rightarrow u(x,y)$ har minimum på C

7 

$f(z) = e^z$, $u(x,y) = e^x \cos y$

$u(x,y)$ har maksimum e for $z = 1$.

$u(x,y)$ har minimum $-e$ for $z = 1 + \pi i$

56

1 $z_n = -2 + i\frac{(-1)^n}{n^2}$ fra eks 2 i afsn 55 (s 183)

$\left|-2+i\frac{(-1)^n}{n^2}-(-2)\right| = \frac{1}{n^2} < \varepsilon$ for $n > \frac{1}{\sqrt{\varepsilon}}$ $\Rightarrow \lim\limits_{n\to\infty} z_n = -2$

2 $z_n = 2 + i\frac{(-1)^n}{n^2}$

$\operatorname{Arg} z_n = \arctan\frac{(-1)^n}{2n^2} \to 0$ for $n\to\infty$, $\Theta \in]-\pi;\pi]$

I eks. 2 afsn 53: Arg z_n har ing. gr. v. for $n \to \infty$

3 $\lim\limits_{n\to\infty} z_n \to z \Rightarrow \lim\limits_{n\to\infty}|z_n| \to |z|$, idet

$\big||z_n|-|z|\big| \le |z_n - z| < \varepsilon$ for $n > N_\varepsilon$ ifølge forudsætning

4 $$\sum_{n=1}^{\infty} z^n = \sum_{n=1}^{\infty}(re^{i\theta})^n = \frac{re^{i\theta}}{1-re^{i\theta}} = \frac{r\cos\theta + ir\sin\theta}{1-r\cos\theta - ir\sin\theta}$$

$$= \frac{-r^2+r\cos\theta+ir\sin\theta}{1+r^2-2r\cos\theta}, \quad 0<r<1$$

$$\Rightarrow \sum_{n=0}^{\infty} r^n\cos n\theta = \frac{r\cos\theta - r^2}{1-2r\cos\theta+r^2} \wedge \sum_{n=0}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1-2r\cos\theta+r^2}$$

59

1 $$z\cosh z^2 = z\sum_{n=0}^{\infty}\frac{(z^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{z^{4n+1}}{(2n)!}$$

2 $f(z) = e^z$, $z_0 = 1$

a $$e^z = \sum_{n=0}^{\infty}\frac{1}{n!}\left.\frac{d^{(n)}e^z}{dz^n}\right|_{z=1}(z-1)^n = \sum_{n=0}^{\infty}\frac{1}{n!}e(z-1)^n$$

$$= e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}$$

b $$e^z = e\,e^{z-1} = e\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!}$$

3 $$f(z) = \frac{z}{z^4+9} = \frac{z}{9}\,\frac{1}{1+\frac{z^4}{9}} = \frac{z}{9}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z^4}{9}\right)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{z^{4n+1}}{9^{n+1}} = \sum_{n=0}^{\infty}(-1)^n\frac{z^{4n+1}}{3^{2n+2}}.$$

59

4 $f(z) = \sin z$

$$f^{(2n)}(z) = (-1)^n \sin z \qquad f^{(2n)}(0) = 0$$

$$f^{(2n+1)}(z) = (-1)^n \cos z \qquad f^{(2n+1)}(0) = (-1)^n$$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = 1 - x + \frac{x^3}{3} - \frac{x^5}{5} + \dots$$

7 $$\frac{1}{1-z} = \frac{1}{1-i-(z-i)} = \frac{1}{1-i}\,\frac{1}{1-\frac{z-i}{1-i}} = \frac{1}{1-i}\sum_{n=0}^{\infty}\left(\frac{z-i}{1-i}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}, \quad |z-i| < \sqrt{2}$$

8 $$\cos z = \sin\left(\frac{\pi}{2} - z\right) = -\sin\left(z - \frac{\pi}{2}\right)$$

$$\cos z = \sum_{n=0}^{\infty} -(-1)^n \frac{(z-\frac{\pi}{2})^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(z-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$$

10 $|z| < \frac{\pi}{2}$, da $\cosh z = 0$ for $z = i\left(\frac{\pi}{2} + p\pi\right)$

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots}{1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots} = z - \frac{z^3}{3} + \dots$$

(polynomium division)

62

2 $$\frac{e^z}{(z+1)^2} = \frac{1}{(z+1)^2}\,\frac{1}{e}\,e^{z+1} = \frac{1}{(z+1)^2}\,\frac{1}{e}\sum_{n=0}^{\infty}\frac{(z+1)^n}{n!}$$

$$= \frac{1}{e}\sum_{n=0}^{\infty}\frac{(z+1)^{n-2}}{n!} = \frac{1}{e}\left(\frac{1}{(z+1)^2} + \frac{1}{z+1} + \sum_{n=2}^{\infty}\frac{(z+1)^{n-2}}{n!}\right)$$

$$= \frac{1}{e}\left(\frac{1}{(z+1)^2} + \frac{1}{z+1} + \sum_{n=0}^{\infty}\frac{(z+1)^n}{(n+2)!}\right), \quad 0 < |z+1| < \infty$$

4 $$f(z) = \frac{1}{z^2(1-z)}, \quad 1 < |z| < \infty$$

i $0 < |z| < 1$

$$f(z) = \frac{1}{z^2}\sum_{n=0}^{\infty} z^n = \frac{1}{z^2} + \frac{1}{z} + \sum_{n=2}^{\infty} z^{n-2} = \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty} z^n$$

ii $1 < |z| < \infty$.

$$f(z) = -\frac{1}{z^3}\,\frac{1}{1-\frac{1}{z}} = -\frac{1}{z^3}\sum_{n=0}^{\infty}\frac{1}{z^n} = -\sum_{n=0}^{\infty}\frac{1}{z^{n+3}} = -\sum_{n=3}^{\infty}\frac{1}{z^n}$$

6 $$\frac{z}{(z-1)(z-3)} = -\frac{1}{2}\frac{1}{z-1} + \frac{3}{2}\frac{1}{z-3} = -\frac{1}{2}\frac{1}{z-1} + \frac{3}{2}\frac{1}{(z-1)-2}$$

$$= -\frac{1}{2}\frac{1}{z-1} - \frac{3}{2^2}\frac{1}{1-\frac{z-1}{2}} = -\frac{1}{2}\frac{1}{z-1} - \frac{3}{2^2}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$$

$$= -\frac{1}{2}\frac{1}{z-1} - 3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} \qquad 0<|z-1|<\infty$$

7 $$f(z) = \frac{1}{z(1+z^2)} = \frac{1}{z} - \frac{z}{1+z^2}$$

i $\quad 0<|z|<1$

$$f(z) = \frac{1}{z} - z\frac{1}{1+z^2} = \frac{1}{z} - z\sum_{n=0}^{\infty}(-1)^n (z^2)^n$$

$$= \frac{1}{z} + \sum_{n=0}^{\infty}(-1)^{n+1} z^{2n+1}$$

ii $\quad 1<|z|<\infty$

$$f(z) = \frac{1}{z} - \frac{1}{z}\frac{1}{1+\left(\frac{1}{z}\right)^2} = \frac{1}{z} - \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{z^2}\right)^n$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}$$

5 $$f(z) = \frac{z+1}{z-1}$$

a $$f(z) = -(z+1)\frac{1}{1-z} = -(z+1)\sum_{n=0}^{\infty} z^n$$

$$= -z\sum_{n=1}^{\infty} z^{n-1} - 1 - \sum_{n=1}^{\infty} z^n$$

$$= -1 - 2\sum_{n=1}^{\infty} z^n \qquad , \qquad |z|<1$$

b $$f(z) = \left(1+\frac{1}{z}\right)\frac{1}{1-\frac{1}{z}} = \left(1+\frac{1}{z}\right)\sum_{n=0}^{\infty}\frac{1}{z^n}$$

$$= 1 + \sum_{n=1}^{\infty}\frac{1}{z^n} + \frac{1}{z}\sum_{n=1}^{\infty}\frac{1}{z^{n-1}}$$

$$= 1 + 2\sum_{n=1}^{\infty}\frac{1}{z^n} \qquad 1<|z|<\infty$$

66

1 $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n , \qquad |z|<1$$

$$\text{diff}_z : \quad -\frac{1}{(1-z)^2}(-1) = \sum_{n=0}^{\infty} n z^{n-1} = \sum_{n=1}^{\infty} n z^{n-1} = \sum_{n=0}^{\infty} (n+1) z^n , \quad |z|<1 \quad *$$

$$\text{diff} : \quad -\frac{2}{(1-z)^3}(-1) = \sum_{n=0}^{\infty} (n+1) n z^{n-1} = \sum_{n=1}^{\infty} (n+1) n z^{n-1} , \quad |z|<1$$

$$\Rightarrow \frac{2}{(1-z)^2} = \sum_{n=0}^{\infty} (n+1)(n+2) z^n , \quad |z|<1$$

2 $z := \frac{1}{1-z}$ i $*$ ovenfor giver

$$\frac{(1-z)^2}{z^2} = \sum_{n=0}^{\infty} \frac{n+1}{(1-z)^n} , \quad \frac{1}{|1-z|}<1 \Leftrightarrow 1<|z-1|$$

$$\Rightarrow \frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{(z-1)^{n+2}} \Rightarrow \frac{1}{z^2} = \sum_{n=2}^{\infty} \frac{(-1)^n (n-1)}{(z-1)^n} , \quad 1<|z-1|$$

3 $$\frac{1}{z} = \frac{1}{2+(z-2)} = \frac{1}{2}\,\frac{1}{1+\frac{z-2}{2}} = \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z-2}{2}\right)^n , \quad \left|\frac{z-2}{2}\right|<1$$

$$\text{diff} : \quad -\frac{1}{z^2} = \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n n \left(\frac{z-2}{2}\right)^{n-1} \frac{1}{2} = \frac{1}{4}\sum_{n=1}^{\infty} (-1)^n n \left(\frac{z-2}{2}\right)^{n-1}$$

$$\Rightarrow \frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty} (-1)^n (n+1) \left(\frac{z-2}{2}\right)^n , \quad |z-2|<2$$

4 $$f(z) = \begin{cases} \frac{\sin z}{z} , & z \neq 0 \\ 1 , & z = 0 \end{cases}$$

$$= \begin{cases} \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) , & z \neq 0 \\ 1 , & z = 0 \end{cases}$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} + \dots , \qquad z \in \mathbb{C}$$

$\Rightarrow f$ kont. $\Rightarrow \lim_{z \to 0} \frac{\sin z}{z} = 1$

5 $$f(z) = \begin{cases} \frac{\cos z}{z^2-(\frac{\pi}{2})^2} , & z \neq \pm\frac{\pi}{2} \\ -\frac{1}{\pi} , & z = \pm\frac{\pi}{2} \end{cases}$$

$$\cos z = 0 - 1\left(z-\tfrac{\pi}{2}\right) + 1\,\tfrac{1}{3!}\left(z-\tfrac{\pi}{2}\right)^3 - \dots \Rightarrow \frac{\cos z}{z-\frac{\pi}{2}} = -1 + \tfrac{1}{3!}\left(z-\tfrac{\pi}{2}\right)^2 - \dots$$

$$f(z) = -\frac{1}{z+\frac{\pi}{2}} + \tfrac{1}{3!}\left(z-\tfrac{\pi}{2}\right)^2 - \dots \Rightarrow f\left(\tfrac{\pi}{2}\right) = -\frac{1}{\pi} \quad (1)$$

$$\cos z = 0 + 1\left(z+\tfrac{\pi}{2}\right) - \tfrac{1}{3!}\left(z+\tfrac{\pi}{2}\right)^3 + \dots \Rightarrow \frac{\cos z}{z+\frac{\pi}{2}} = 1 - \tfrac{1}{3!}\left(z+\tfrac{\pi}{2}\right)^2 + \dots$$

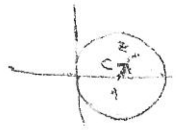
$$f(z) = \frac{1}{z-\frac{\pi}{2}} - \tfrac{1}{3!}\left(z+\tfrac{\pi}{2}\right)^2 + \dots \Rightarrow f\left(-\tfrac{\pi}{2}\right) = -\frac{1}{\pi} \quad (2)$$

(1), (2) $\Rightarrow f$ er analytisk i $\pm\frac{\pi}{2}$

f er hel

66

6 $\frac{1}{w} = \sum_{n=0}^{\infty} (-1)^n (w-1)^n \quad , \quad |w-1| < 1$



C : kurve fra $w=1$ til $w=z$

int. : $\quad \mathrm{Log}\, z = \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n+1}}{n+1} \quad , \quad |z-1| < 1$

$\quad = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n \quad , \quad |z-1| < 1$

7 $f(z) = \begin{cases} \frac{\mathrm{Log}\, z}{z-1} & , z \neq 1 \\ 1 & , z = 1 \end{cases}$

analytisk?

$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^{n-1} \quad , \quad |z-1| < 1 \quad ,$ dvs kont i $z=1$

Log z analytisk i $0 < |z|$, $-\frac{\pi}{2} < \mathrm{Arg}\, z < \frac{\pi}{2}$

$z-1$ analytisk i $\mathbb{C}$

$\Rightarrow f(z)$ analytisk i $0 < |z|$, $-\frac{\pi}{2} < \mathrm{Arg}\, z < \frac{\pi}{2}$

8 f analytisk i z_0

$f(z_0) = f'(z_0) = \dots = f^{(m)}(z_0) = 0$ $\quad \Big| \quad g(z) = \begin{cases} \frac{f(z)}{(z-z_0)^{m+1}} & , z \neq z_0 \\ \frac{f^{(m+1)}(z_0)}{(m+1)!} & , z = z_0 \end{cases}$

$f(z) = \sum_{n=m+1}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n = \frac{f^{(m+1)}(z_0)}{(m+1)!} (z-z_0)^{m+1} + \frac{f^{(m+2)}(z_0)}{(m+2)!} (z-z_0)^{m+2} + \dots$

$g(z) = \frac{f^{(m+1)}(z_0)}{(m+1)!} + \frac{f^{(m+2)}(z_0)}{(m+2)!} (z-z_0) + \dots$

g udviklet i Taylorrække om z_0 $\Rightarrow$ g analytisk i z_0

11 $f_1(z) = \frac{1}{z^2+1} = \sum_{n=0}^{\infty} (-1)^n z^{2n} \quad , \quad |z^2| < 1 \Leftrightarrow |z| < 1$

$f_2(z) = \frac{1}{z^2+1} \quad , \quad z \neq \pm i$, er analytisk, da z^2+1 er analytisk

$f_2(z)$ er analytisk fortsættelse af $f_1(z)$ ind i $\mathbb{C} \setminus \{\pm i\}$, da f_1 og f_2 stemmer overens overalt i $\mathbb{C} \setminus \{\pm i\}$

67

1 $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad , \quad \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n} \quad |z| < 1$

$\frac{e^z}{z(1+z^2)} = \frac{1}{z} \left(1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \dots \right) \left(1 - z^2 + z^4 - \dots \right)$

$\quad = \frac{1}{z} + 1 - \frac{1}{2} z - \frac{5}{6} z^2 + \frac{13}{24} z^3 + \dots \quad , \quad |z| < 1$

2 $$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad z \in \mathbb{C}$$

$$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \;\Big|\; 1 \;\Big|\; \frac{1}{z} + \frac{z}{3!} + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right) z^3 + \dots = \frac{1}{z} + \frac{1}{6} z + \frac{7}{360} z^3 + \dots$$

$$1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

$$\frac{z^2}{3!} - \frac{z^4}{5!} + \dots$$

$$\frac{z^2}{3!} - \frac{z^4}{(3!)^2} + \dots$$

$$\left(\frac{1}{(3!)^2} - \frac{1}{5!}\right) z^4 - \dots$$

$$\left(\frac{1}{(3!)^2} - \frac{1}{5!}\right) z^4 - \dots$$

$$\frac{1}{\sin z} = \frac{1}{z} + \frac{z}{3!} + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right) z^3 + \dots, \quad 0 < |z| < \pi$$

3 $$e^z - 1 = \sum_{n=0}^{\infty} \frac{z^n}{n!} - 1, \quad e^z - 1 = 0 \iff z = p2\pi i, \; p \in \mathbb{Z}$$

$$z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120} \;\Big|\; 1 \;\Big|\; \frac{1}{z} - \frac{1}{2} + \frac{z}{12} - \frac{z^3}{720} + \dots$$

$$1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{24} + \frac{z^4}{120} + \dots$$

$$-\frac{z}{2} - \frac{z^2}{6} - \frac{z^3}{24} - \frac{z^4}{120} + \dots$$

$$-\frac{z}{2} - \frac{z^2}{4} - \frac{z^3}{12} - \frac{z^4}{48} + \dots$$

$$\frac{z^2}{12} + \frac{z^3}{24} + \frac{z^4}{80} + \dots$$

$$\frac{z^2}{12} + \frac{z^3}{24} + \frac{z^4}{72} + \dots$$

$$-\frac{z^4}{720} - \dots$$

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \frac{z}{12} - \frac{z^3}{720} + \dots, \quad 0 < |z| < 2\pi$$

4 $$\frac{1}{z^2 \sinh z} = \frac{1}{z^3} - \frac{1}{6}\frac{1}{z} + \frac{7}{360} z + \dots, \quad C:$$

$$\int_C \frac{dz}{z^2 \sinh z} = 2\pi i \, b_1 = 2\pi i \left(-\frac{1}{6}\right) = -\frac{1}{3}\pi i$$

6 Vis $$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}.$$

Induktion:

$n = 1$: $(fg)' = fg' + f'g$ ok

fortsættes

6 fortsatt

$n = m$: antar $(fg)^{(m)} = \sum_{k=0}^{m} \binom{m}{k} f^{(k)} g^{(m-k)}$ gyldig

$$(fg)^{(m+1)} = (fg' + f'g)^{(m)}$$
$$= \sum_{k=0}^{m} \binom{m}{k} f^{(k)} g^{(m-k+1)} + \sum_{k=0}^{m} \binom{m}{k} f^{(k+1)} g^{(m-k)}$$
$$= f g^{(m+1)} + \sum_{k=1}^{m} \binom{m}{k} f^{(k)} g^{(m+1-k)}$$
$$+ \sum_{k=0}^{m-1} \binom{m}{k} f^{(k+1)} g^{(m-k)} + f^{(m+1)} g$$
$$= f g^{(m+1)} + \sum_{k=1}^{m} \left[\binom{m}{k} + \binom{m}{k-1} \right] f^{(k)} g^{(m+1-k)} + f^{(m+1)} g$$
$$= \sum_{k=0}^{m+1} \binom{m+1}{k} f^{(k)} g^{(m+1-k)} \quad \text{ok}$$

7 $f(z) = z + a_2 z^2 + a_3 z^3 + \dots, \quad z \in \mathbb{C}$

a $g(z) = f \circ f(z)$

$$g'(z) = [1 + 2a_2(z + a_2 z^2 + a_3 z^3 + \dots)$$
$$+ 3a_3(z + a_2 z^2 + a_3 z^3 + \dots)$$
$$= 1 + 4a_2 z + (6a_2^2 + 6a_3) z^2 + \dots$$
$$g(z) = z + 2a_2 z^2 + 2(a_2^2 + a_3) z^3 + \dots$$

b (formelt)

$$f \circ f(z) = z + a_2 z^2 + a_3 z^3 + a_2 (z + a_2 z^2 + \dots)^2$$
$$+ a_3 (z + a_2 z^2 + \dots)^3$$
$$= z + 2a_2 z^2 + 2(a_2^2 + a_3) z^3 + \dots$$

c $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots, \quad z \in \mathbb{C}$

$$\sin \circ \sin z = z + 0 z^2 + 2\left(0^2 - \frac{1}{6}\right) z^3 + \dots$$
$$= z - \frac{1}{3} z^3 + \dots, \quad z \in \mathbb{C}$$

8 $\frac{1}{\cosh z} = \sum_{n=0}^{\infty} \frac{E_n}{n!} z^n, \quad |z| < \frac{\pi}{2}$ $\quad (\cosh z = 0$ for $z = i(\frac{\pi}{2} + k\pi))$

$\frac{1}{\cosh z}$ like $\Rightarrow E_{2n+1} = 0$

fortsettes

8 fortsatt

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \dots$$

$$1 + \frac{z^2}{2} + \frac{z^4}{24} + \frac{z^6}{720} + \dots \Big|\ 1 \qquad \Big|\ 1 - \frac{z^2}{2} + \frac{5}{24}z^4 - \frac{61}{720}z^6 + \dots$$

$$1 + \frac{z^2}{2} + \frac{z^4}{24} + \frac{z^6}{720} + \dots$$

$$- \frac{z^2}{2} - \frac{z^4}{24} - \frac{z^6}{720} - \dots$$

$$- \frac{z^2}{2} - \frac{z^4}{4} - \frac{z^6}{48} - \dots$$

$$\frac{5}{24}z^4 + \frac{7}{360}z^6 + \dots$$

$$\frac{5}{24}z^4 + \frac{5}{48}z^6 + \dots$$

$$- \frac{61}{720}z^6 - \dots$$

$$\frac{1}{\cosh z} = 1 - \frac{1}{2}z^2 + \frac{5}{24}z^4 - \frac{61}{720}z^6 + \dots$$

$$\Rightarrow \quad E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61$$

71

1 a $\dfrac{1}{z+z^2} = \dfrac{1}{z}\dfrac{1}{1+z} = \dfrac{1}{z}(1-z+\dots) = \dfrac{1}{z} - 1 + \dots$, $B = 1$

b $z\cos\dfrac{1}{z} = z\left(1 - \dfrac{1}{2z^2} + \dfrac{1}{24z^4} + \dots\right) = z - \dfrac{1}{2z} + \dfrac{1}{24z^3} + \dots$, $B = -\dfrac{1}{2}$

c $\dfrac{z-\sin z}{z} = \dfrac{1}{z}\left(z - \left(z - \dfrac{z^3}{3!} + \dots\right)\right) = \dfrac{z^2}{3!} - \dfrac{z^4}{5!} + \dots$, $B = 0$

d $\dfrac{\cot z}{z^4} = \dfrac{1}{z^4}\left(\dfrac{1}{z} - \dfrac{z}{3} - \dfrac{z^3}{45} - \dfrac{2z^5}{945} - \dots\right) = \dfrac{1}{z^5} - \dfrac{1}{3z^3} - \dfrac{1}{45z} - \dfrac{2z}{945} - \dots$

$B = -\dfrac{1}{45}$

e $\dfrac{\sinh z}{z^4(1-z^2)} = \dfrac{1}{z^4}\left(z + \dfrac{z^3}{3!} + \dots\right)\left(1 + z^2 + \dots\right) = \dfrac{1}{z^3} + \dfrac{7}{6}\dfrac{1}{z} + \dots$,

$B = \dfrac{7}{6}$

2 $C: |z| = 3$

a $\dfrac{\exp(-z)}{z^2} = \dfrac{1}{z^2}\left(1 - z + \dfrac{z^2}{2} - \dots\right) = \dfrac{1}{z^2} - \dfrac{1}{z} + \dfrac{1}{2} - \dots$, $B = -1$

$\int_C \dfrac{\exp(-z)}{z^2}\,dz = 2\pi i(-1) = -2\pi i$

b $\dfrac{\exp(-z)}{(z-1)^2} = \dfrac{1}{(z-1)^2}\left(\dfrac{1}{e} - \dfrac{1}{e}(z-1) + \dfrac{1}{e}(z-1)^2 - \dots\right)$

$= \dfrac{1}{e(z-1)^2} - \dfrac{1}{e}\dfrac{1}{z-1} + \dfrac{1}{e} - \dots$, $B = -\dfrac{1}{e}$

$\int_C \dfrac{\exp(-z)}{(z-1)^2}\,dz = 2\pi i\left(-\dfrac{1}{e}\right) = -\dfrac{2\pi}{e}i$

c $z^2\exp\dfrac{1}{z} = z^2\left(1 + \dfrac{1}{z} + \dfrac{1}{2z^2} + \dfrac{1}{6z^3} + \dfrac{1}{24z^4}\right.$

$= z^2 + z + \dfrac{1}{2} + \dfrac{1}{6}\dfrac{1}{z} + \dfrac{1}{24}\dfrac{1}{z^2} + \dots$, $B = \dfrac{1}{6}$

$\int_C z^2\exp\dfrac{1}{z}\,dz = 2\pi i\cdot\dfrac{1}{6} = \dfrac{1}{3}\pi i$

d $\dfrac{z+1}{z^2-2z} = \dfrac{z+1}{z(z-2)} = -\dfrac{1}{2z} + \dfrac{3}{2(z-2)}$, $B_1 = -\dfrac{1}{2}$, $B_2 = \dfrac{3}{2}$

$\int_C \dfrac{z+1}{z^2-2z}\,dz = 2\pi i\left(-\dfrac{1}{2} + \dfrac{3}{2}\right) = 2\pi i$

4 $f(z) = \exp\left(z + \dfrac{1}{z}\right) = \exp z \exp\dfrac{1}{z} = \sum_{n=0}^{\infty}\dfrac{z^n}{n!}\sum_{k=0}^{\infty}\dfrac{1}{k!\,z^k}$

$= \dots + \sum_{n=0}^{\infty}\dfrac{1}{n!(n+1)!}\dfrac{1}{z} + \dots$, $B = \sum_{n=0}^{\infty}\dfrac{1}{n!(n+1)!}$

$C: |z| = 1$, $\int_C f(z)\,dz = 2\pi i\sum_{n=0}^{\infty}\dfrac{1}{n!(n+1)!}$

2 a
$$\frac{1-\cosh z}{z^3} = \frac{1}{z^3}\left(1-\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\dots\right)\right) = -\frac{1}{2z} - \frac{z}{24} - \dots$$
$m=1, \quad B=-\frac{1}{2}$

b
$$\frac{1-\exp(2z)}{z^4} = \frac{1}{z^4}\left(1-\left(1+2z+\frac{(2z)^2}{2!}+\frac{(2z)^3}{3!}+\frac{(2z)^4}{4!}\right.\right.$$
$$= \frac{1}{z^4}\left(-2z-4z^2-\frac{4}{3}z^3-\frac{2}{3}z^4-\dots\right.$$
$$= -\frac{2}{z^3}-\frac{4}{z^2}-\frac{4}{3z}-\frac{2}{3}-\dots$$
$m=3, \quad B=-\frac{4}{3}$

c
$$\frac{\exp(2z)}{(z-1)^2} = \frac{1}{(z-1)^2}\left(1+2e^2(z-1)+4e^2(z-1)^2+\dots\right)$$
$$= \frac{1}{(z-1)^2}+\frac{2e^2}{z-1}+4e^2+\dots$$
$m=2, \quad B=2e^2$

3 $f(z)$ analytisk i z_0, $\quad g(z)=\frac{f(z)}{z-z_0}$
$$g(z)=\frac{1}{z-z_0}\left(f(z_0)+f'(z_0)(z-z_0)+\frac{1}{2}f''(z_0)(z-z_0)^2+\dots\right.$$
$$= \frac{f(z_0)}{z-z_0}+f'(z_0)+\frac{1}{2}f''(z_0)(z-z_0)+\dots$$

a $f(z_0)\neq 0$

g har en simpel pol i z_0, $\quad B=f(z_0)$

b $f(z_0)=0$

g har en hævelig singularitet i z_0

5
$$f(z)=\frac{8a^3z^2}{(z^2+a^2)^3}=\frac{8a^3z^2}{(z+ai)^3(z-ai)^3}=\frac{\varphi(z)}{(z-ai)^3}$$
$$\varphi(z)=\frac{8a^3z^2}{(z+ai)^3} \quad \text{analytisk i} \quad 0<|z+ai|$$
$$\varphi(ai)=\frac{8a^3(ai)^2}{(2ai)^3}=\frac{8a^3}{8ai}=-a^2i$$
$$\varphi'(z)=\frac{(z+ai)^3\,16a^3z-8a^3z^2\,3(z+ai)^2}{(z+ai)^6}$$
$$=\frac{16a^3z(z+ai)-24a^3z^2}{(z+ai)^4}=\frac{-8a^3z^2+16a^4zi}{(z+ai)^4}$$

fortsættes

72

5 fortsat

$$q'(ai) = \frac{8a^5 - 16a^5}{16a^4} = -\frac{a}{2}$$

$$q''(z) = \frac{(z+ai)^4(-16a^3z + 16a^4 i) - (-8a^3z^2 + 16a^4 i z)\,4(z+ai)^3}{(z+ai)^8}$$

$$= \frac{(z+ai)(-16a^3)(z-ai) + 4(8a^3z^2 - 16a^4 z i)}{(z+ai)^5}$$

$$q''(ai) = \frac{4(-8a^5 + 16a^5)}{32a^5 i} = \frac{8a^5}{8a^5 i} = -i$$

principaldelen af $f(z)$:

$$\frac{1}{(z-ai)^3}\left(-a^2 i - \frac{a}{2}(z-ai) - \frac{i}{2}(z-ai)^2\right)$$

$$= -\frac{i}{2(z-ai)} - \frac{a}{2(z-ai)^2} - \frac{a^2 i}{(z-ai)^3}$$

74

1 a $\quad \frac{z^2+2}{z-1}$, $m = 1$, $B = 1 + 2 = 3$

b $\quad \left(\frac{z}{2z+1}\right)^3 = \frac{z^3}{8(z+\frac{1}{2})^3}$, $m = 3$, $B = \frac{3 \cdot 2\,(-\frac{1}{2})}{8 \cdot 2} = -\frac{3}{16}$

c $\quad \frac{\exp z}{z^2+\pi^2} = \frac{\exp z}{(z+i\pi)(z-i\pi)}$, $m = 1$, $B = \frac{-1}{\pm 2i\pi} = \mp \frac{i}{2\pi}$

2 a $\quad \frac{z^{\frac{1}{4}}}{z+1}$, $|z| > 0$, $0 < \arg z < 2\pi$

$m = 1$, $B = (-1)^{\frac{1}{4}} = \exp\left(\frac{1}{4}\log(-1)\right) = \exp\frac{i\pi}{4} = \frac{1}{\sqrt{2}}(1+i)$

b $\quad \frac{\operatorname{Log} z}{(z^2+1)^2} = \frac{\operatorname{Log} z}{(z+i)^2(z-i)^2}$, $|z| > 0$, $-\pi < \operatorname{Arg} z < \pi$

$z_0 = i$, $m = 2$, $q(z) = \frac{\operatorname{Log} z}{(z+i)^2}$

$$q'(z) = \frac{(z+i)^2\,\frac{1}{z} - \operatorname{Log} z \cdot 2(z+i)}{(z+i)^4}$$

$$B = \frac{-4\,\frac{1}{i} - i\,\frac{\pi}{2} \cdot 2i}{16} = \frac{2\pi + 4i}{16} = \frac{\pi + 2i}{8}$$

fortsættes

2 fortsat

c $\dfrac{z^{\frac12}}{(z^2+1)^2} = \dfrac{z^{\frac12}}{(z+i)^2(z-i)^2}$, $|z|>0$, $0 < \arg z < 2\pi$

$z_0 = i$, $m = 2$, $\varphi(z) = \dfrac{z^{\frac12}}{(z+i)^2}$

$\varphi'(z) = \dfrac{(z+i)^2 \frac12 z^{-\frac12} - z^{\frac12}\, 2(z+i)}{(z+i)^4}$

$B = \varphi'(i) = \dfrac{-4\frac12 \exp(-\frac12 i \frac{\pi}{2}) - \exp(\frac12 i \frac{\pi}{2})\, 2i}{16}$

$= \dfrac{-(1-i)-(1+i)2i}{8\sqrt2} = \dfrac{1-i}{8\sqrt2} = \dfrac{\sqrt2}{16}(1-i)$

4 $\displaystyle\int_C \frac{dz}{z^3(z+4)}$

a $C: |z| = 2$, $z_0 = 0$

$m = 3$, $\varphi(z) = \dfrac{1}{z+4}$, $\varphi'(z) = -\dfrac{1}{(z+4)^2}$, $\varphi''(z) = \dfrac{2}{(z+4)^3}$

$\displaystyle\int_C \frac{dz}{z^3(z+4)} = 2\pi i \,\frac12\, \frac{2}{4^3} = \frac{\pi i}{32}$

b $C: |z+2| = 3$

$z_0 = 0$, $B_1 = \dfrac{1}{64}$ jf spm. a

$z_0 = -4$, $m = 1$, $B_2 = \dfrac{1}{(-4)^3} = -\dfrac{1}{64}$

$\displaystyle\int_C \frac{dz}{z^3(z+4)} = 2\pi i \left(\frac{1}{64} - \frac{1}{64}\right) = 0$

5 $\displaystyle\int_C \frac{\cosh \pi z}{z(z^2+1)}\, dz = \int_C \frac{\cosh \pi z}{z(z+i)(z-i)}\, dz$, $C: |z| = 2$

$= 2\pi i \left(\dfrac{1}{i(-i)} + \dfrac{\cosh(-\pi i)}{-i(-2i)} + \dfrac{\cosh \pi i}{i \cdot 2i} \right)$

$= 2\pi i \left(1 + \dfrac{-1}{-2} + \dfrac{-1}{-2} \right) = 4\pi i$

6 $C: |z| = 3$

a $f(z) = \dfrac{(3z+2)^2}{z(z-1)(2z+5)}$

$\dfrac{1}{z^2} f\left(\dfrac1z\right) = \dfrac{1}{z^2} \dfrac{\left(\frac3z + 2\right)^2}{\frac1z\left(\frac1z - 1\right)\left(\frac2z + 5\right)} = \dfrac1z \dfrac{(3+2z)^2}{(1-z)(2+5z)}$

$\displaystyle\int_C f(z)\, dz = 2\pi i \cdot \frac{3^2}{1\cdot 2} = 9\pi i$

fortsættes

74

6 fortsat

b $$f(z) = \frac{z^3(1-3z)}{(1+z)(1+2z^4)}$$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \frac{\frac{1}{z^3}\left(1-\frac{3}{z}\right)}{\left(1+\frac{1}{z}\right)\left(1+\frac{2}{z^4}\right)} = \frac{1}{z} \frac{z-3}{(z+1)(z^4+2)}$$

$$\int_C f(z)\,dz = 2\pi i \frac{-3}{1 \cdot 2} = -3\pi i$$

c $$f(z) = \frac{z^3 e^{\frac{1}{z}}}{1+z^3}$$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \frac{\frac{1}{z^3} e^z}{1+\frac{1}{z^3}} = \frac{1}{z^2} \frac{e^z}{z^3+1}, \quad m = 2$$

$$g(z) = \frac{e^z}{z^3+1}, \quad g'(z) = \frac{(z^3+1)e^z - e^z 3z^2}{(z^3+1)^2}$$

$$\int_C f(z)\,dz = 2\pi i \frac{1-0}{1} = 2\pi i$$

76

2 a $$\frac{z-\sinh z}{z^2 \sinh z} = \frac{1}{z \sinh z} - \frac{1}{z^2}; \quad \frac{1}{z^2}: B_2 = 0$$

$$\frac{1}{z \sinh z}: \quad p(z) = 1 \neq 0, \quad q(z) = z \sinh z, \quad q(\pi i) = 0$$

$$q'(z) = \sinh z + z \cosh z, \quad q'(\pi i) = 0 + \pi i(-1) = -\pi i$$

$$B_1 = \frac{1}{-\pi i} = \frac{i}{\pi}$$

$$B = \frac{i}{\pi} - 0 = \frac{i}{\pi}$$

b $$\frac{\exp(zt)}{\sinh z} = \frac{p(z)}{q(z)}, \quad p(\pm\pi i) = \exp(\pm\pi t i) \neq 0$$

$$q(\pm\pi i) = \sinh(\pm\pi i) = 0, \quad q'(z) = \cosh z$$

$$q'(\pm\pi i) = \cosh(\pm\pi i) = \cos(\pm\pi) = -1$$

$$B = \frac{\exp(\pi t i)}{-1} + \frac{\exp(-\pi t i)}{-1} = -2\cos(\pi t)$$

3 a $$\frac{z}{\cos z} = \frac{p(z)}{q(z)}, \quad z_n = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}, \quad p(z_n) \neq 0$$

$$q(z_n) = 0, \quad q'(z) = -\sin z, \quad q'(z_n) = -(-1)^n = (-1)^{n+1}$$

$$B = \frac{z_n}{(-1)^{n+1}} = (-1)^{n+1} z_n$$

fortsættes

3 fortsat

b $\quad \tanh z = \frac{\sinh z}{\cosh z} = \frac{p(z)}{q(z)}, \quad z_n = i\left(\frac{\pi}{2} + n\pi\right), \quad p(z_n) \neq 0$

$$q(z_n) = 0, \quad q'(z) = \sinh z$$

$$B = \frac{p(z_n)}{q'(z_n)} = \frac{\sinh z_n}{\sinh z_n} = 1$$

4 $\quad C: |z| = 2$

a $\quad \int_C \tan z \, dz = 2\pi i \cdot 2(-1) = -4\pi i$, jf. opg. 3 b,

her dog $B = -1$, da $\cos' = -\sin$

b $\quad \int_C \frac{dz}{\sinh 2z} = \int_C \frac{p(z)}{q(z)} dz, \quad p(z) = 1 \neq 0$

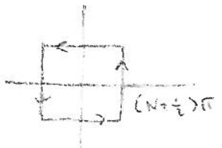
$$q(z) = \sinh 2z, \quad q\left(\pm i\frac{\pi}{2}\right) = \sinh(\pm i\pi) = 0, \quad q(0) = 0$$

$$q'(z) = 2\cosh 2z, \quad q'\left(\pm i\frac{\pi}{2}\right) = 2\cosh(\pm i\pi) = 2\cos(\pm\pi) = -2,$$

$$q'(0) = 2$$

$$\int_C \frac{dz}{\sinh 2z} = 2\pi i \cdot \left(-\frac{1}{2} + \frac{1}{2} - \frac{1}{2}\right) = -\pi i$$

5 $\quad \int_{C_N} \frac{dz}{z^2 \sin z}, \qquad C:$



B_0: $$\frac{1}{z^2 \sin z} = \frac{1}{z^2}\left(\frac{1}{z} + \frac{1}{3!}z + \left(\frac{1}{3!} - \frac{1}{5!}\right)z^3 + \ldots\right.$$

$$= \frac{1}{z^3} + \frac{1}{3!}\frac{1}{z} + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right)z + \ldots \Rightarrow B_0 = \frac{1}{6}$$

B_n: $$\frac{1}{z^2 \sin z} = \frac{p(z)}{q(z)}, \quad p(z) = 1 \neq 0$$

$$q(z) = 0 \perp z_n = n\pi, \quad n \in \mathbb{Z}_+$$

$$q'(z) = 2z\sin z + z^2\cos z, \quad q'(z_n) = 0 + (n\pi)^2(-1)^n$$

$$B_n = (-1)^n (n\pi)^2$$

$$\int_C \frac{dz}{z^2 \sin z} = 2\pi i \left(\frac{1}{6} + \frac{2}{\pi^2}\sum_{n=1}^{N}\frac{(-1)^n}{n^2}\right)$$

fortsættes

5 fortsat

$$\left| \int_C \frac{dz}{z^2 \sin z} \right| \le \frac{16}{(2N+1)\pi} \quad (\text{se * nedenfor})$$

$$\to 0 \text{ for } N \to \infty$$

$$\Rightarrow \frac{1}{6} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = 0 \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

6 $$\int_C \frac{dz}{(z^2-1)^2+3} = \int_C \frac{p(z)}{q(z)}\, dz, \quad C: $$  $, \quad p(z) = 1 \ \forall z,$

$q(z) = 0$ for $(z^2-1)^2+3 = z^4 - 2z^2 + 4 = 0 \quad d = 4 - 16 = -12$

$$\Leftrightarrow z^2 = \frac{2 \pm i\sqrt{12}}{2} = 1 \pm i\sqrt{3} \Leftrightarrow z = \begin{cases} \pm \frac{1}{2}(\sqrt{6} + i\sqrt{2}) \\ \pm \frac{1}{2}(\sqrt{6} - i\sqrt{2}) \end{cases}$$

inden for C: $z_0 = \frac{1}{2}(\sqrt{6} + i\sqrt{2})$ og $-\bar{z}_0 = \frac{1}{2}(-\sqrt{6} + i\sqrt{2})$

$$q'(z) = 2(z^2-1)\,2z = 4z(z^2-1)$$

$$q'(z_0) = 4 \cdot \tfrac{1}{2}(\sqrt{6} + i\sqrt{2})\, i\sqrt{3} = -2\sqrt{6} + 6\sqrt{2}\, i = B_1$$

$$q'(-\bar{z}_0) = 4\, \tfrac{1}{2}(\sqrt{2} - i\sqrt{3})\, i\sqrt{3} = 2\sqrt{6} + 6\sqrt{2}\, i = B_2$$

$$\frac{1}{B_1} + \frac{1}{B_2} = \frac{-2\sqrt{6} - 6\sqrt{2}\, i}{96} + \frac{2\sqrt{6} - 6\sqrt{2}\, i}{96} = \frac{-12\sqrt{2}\, i}{96} = -\frac{\sqrt{2}}{8} i$$

$$\int_C \frac{dz}{(z^2-1)+3} = 2\pi i \left(-\frac{\sqrt{2}}{8} i\right) = \frac{\pi\sqrt{2}}{4}$$

* $$|\sin z|^2 = \sin^2 x + \sinh^2 y \Rightarrow \begin{cases} |\sin z| \ge |\sin x| \\ |\sin z| \ge |\sinh y| \end{cases}$$

på lodrette sider: $|\sin z| \ge \sin |(N+\frac{1}{2})\pi| = \sin \frac{\pi}{2} = 1$

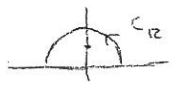
på vandrette side: $|\sin z| \ge \sinh (N+\frac{1}{2})\pi \ge \sinh \frac{\pi}{2}$

$\Rightarrow |\sin z| \ge 1$ på C $\quad (\sinh \frac{\pi}{2} \approx 2{,}3013)$

$$\left| \int_C \frac{dz}{z^2 \sin z} \right| \le \int_C \frac{1}{(\sqrt{2}(N+\frac{1}{2})\pi)^2 \cdot 1}\, dz = \frac{4 \cdot 2(N+\frac{1}{2})\pi}{(\sqrt{2}(N+\frac{1}{2})\pi)^2}$$

$$= \frac{16}{(2N+1)\pi}$$

1 $\quad f(z) = \dfrac{1}{z^2+1} = \dfrac{1}{(z+i)(z-i)}$



f lige

$\varphi(z) = \dfrac{1}{z+i}$, $\varphi(i) = \dfrac{1}{2i}$; $\left| \int_{C_R} f(z)\,dz \right| \le \dfrac{1}{R^2-1} \pi R \to 0$ for $R \to \infty$

$$\int_0^\infty \frac{1}{x^2+1}\,dx = \frac{1}{2}\, 2\pi i\, \frac{1}{2i} = \frac{\pi}{2}$$

2 $\quad f(z) = \dfrac{1}{(z^2+1)^2} = \dfrac{1}{(z+i)^2(z-i)^2}$

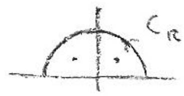
f lige

$\varphi(z) = \dfrac{1}{(z+i)^2}$, $\varphi'(z) = -\dfrac{2}{(z+i)^3}$, $\varphi'(i) = -\dfrac{2}{(2i)^3} = \dfrac{1}{4i}$

$\left| \int_{C_R} f(z)\,dz \right| \le \dfrac{1}{(R^2-1)^2} \pi R \to 0$ for $R \to \infty$

$$\int_0^\infty \frac{1}{(x^2+1)^2}\,dx = \frac{1}{2}\, 2\pi i\, \frac{1}{4i} = \frac{\pi}{4}$$

3 $\quad f(z) = \dfrac{1}{z^4+1} = \dfrac{p(z)}{q(z)}$



f lige

$q(z) = 0$ for $z = \frac{1}{\sqrt{2}}(\pm 1 + i)$

$p(z) = 1$, $q'(z) = 4z^3$

$$B_1 + B_2 = \frac{1}{4\left(\frac{1}{\sqrt2}(1+i)\right)^3} + \frac{1}{4\left(\frac{1}{\sqrt2}(-1+i)\right)^3} = \frac{2\sqrt2}{4\cdot 2i}\left(\frac{1}{1+i} - \frac{1}{-1+i}\right)$$
$$= \frac{\sqrt2}{4i}\,\frac{-2}{-2} = \frac{\sqrt2}{4i}$$

$\left| \int_{C_R} f(z)\,dz \right| \le \dfrac{1}{R^4-1} \pi R \to 0$ for $R \to \infty$

$$\int_0^\infty \frac{1}{x^4+1}\,dx = \frac{1}{2}\, 2\pi i\, \frac{\sqrt2}{4i} = \frac{\pi\sqrt2}{4}$$

4 $\quad f(z) = \dfrac{z^2}{(z^2+1)(z^2+4)} = \dfrac{p(z)}{q(z)}$

f lige

$q(z) = 0$ for $z = \begin{cases} i \\ 2i \end{cases}$

$q'(z) = 2z(z^2+1) + 2z(z^2+4) = 2z(2z^2+5)$

$p(i) = -1$, $q'(i) = 2i(-2+5) = 6i$ $\qquad B_1 = -\dfrac{1}{6i}$

$p(2i) = -4$, $q'(2i) = 4i(-8+5) = -12i$ $\qquad B_2 = \dfrac{1}{3i}$

$\left| \int_{C_R} f(z)\,dz \right| \le \dfrac{R^2}{(R^2-1)(R^2-4)} \pi R \to 0$ for $R \to \infty$

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \frac{1}{2}\, 2\pi i \left(-\frac{1}{6i} + \frac{1}{3i}\right) = \frac{\pi}{6}$$

5 $f(z) = \dfrac{z^2}{(z^2+9)(z^2+4)^2} = \dfrac{z^2}{(z+3i)(z-3i)(z+2i)^2(z-2i)^2}$, f lige

(1) $\varphi(z) = \dfrac{z^2}{(z+3i)(z^2+4)^2}$

$\varphi(3i) = \dfrac{-9}{6i\,(-5)^2} = -\dfrac{3}{50i} = B_1$

(2) $\varphi(z) = \dfrac{z^2}{(z^2+9)(z+2i)^2}$

$\varphi'(z) = \dfrac{(z^2+9)(z+2i)^2\,2z - z^2\left(2z(z+2i)^2 + (z^2+9)\,2(z+2i)\right)}{(z^2+9)^2(z+2i)^4}$

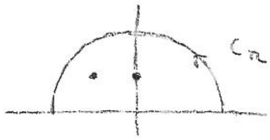
$\varphi'(2i) = \dfrac{5(-16)4i - (-4)(4i(-16) + 10\cdot 4i}{25\cdot 256} = \dfrac{-20i-6i}{25\cdot 16} = -\dfrac{13i}{200} = B_{2i}$

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{R^2}{(R^2-9)(R^2-4)^2}\,\pi R \to 0 \quad \text{for } R\to\infty$$

$$\int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)^2}\,dx = \frac{1}{2}\,2\pi i\left(-\frac{3}{50i} - \frac{13i}{200}\right) = \frac{\pi}{200}$$

7 $f(z) = \dfrac{z}{(z^2+1)(z^2+2z+2)}$, $\quad z^2+1 = 0$ for $z = \pm i$

$z^2+2z+2 = 0$ f $z = -1 \pm 2i$



(1) $\varphi(z) = \dfrac{z}{(z+i)(z^2+2z+2)}$

(2) $\varphi(z) = \dfrac{z}{(z^2+1)(z+1+2i)}$

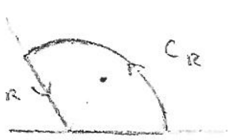
$B_1 + B_2 = \dfrac{i}{2i(2i+1)} + \dfrac{-1+i}{(-2i+1)2i} = \dfrac{i(1-2i) + (-1+i)(2i+1)}{2i\cdot 5}$

$= -\dfrac{1}{10i}$

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{R}{(R^2-1)(R-\sqrt{2})^2}\,\pi R \to 0 \quad \text{for } R\to\infty$$

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2x+2)}\,dx = 2\pi i\left(-\frac{1}{10i}\right) = -\frac{\pi}{5}$$

8 $f(z) = \dfrac{1}{z^3+1} = \dfrac{p(z)}{q(z)}$



$q(z) = 0$ for $z = \begin{cases} e^{\pm i\frac{\pi}{3}} \\ -1 \end{cases}$

$p(z) = 1$

$q'(z) = 3z^2$, $\quad q'(e^{i\frac{\pi}{3}}) = 3e^{i\frac{2\pi}{3}}$, $\quad B = \frac{1}{3}e^{-i\frac{2\pi}{3}} = \frac{1}{3}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= -\frac{1}{6}\left(1 + i\sqrt{3}\right)$

fortsættes

79

8 fortsat

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{1}{R^3-1}\,\frac{2}{3}\pi R \to 0 \quad \text{for} \quad R\to\infty$$

L_R: $z = re^{i\frac{2\pi}{3}}$, $dz = e^{i\frac{2\pi}{3}}\,dr$

$$\int_{L_R} f(z)\,dz = -\int_0^R \frac{e^{i\frac{2\pi}{3}}}{r^3 e^{i2\pi}+1}\,dr = -\left(-\frac12 + i\frac{\sqrt3}{2}\right)\int_0^R \frac{1}{r^3+1}\,dr$$

$$\int_0^\infty \frac{1}{x^3+1}\,dx + 0 + \left(\frac12 - i\frac{\sqrt3}{2}\right)\int_0^\infty \frac{1}{r^3+1}\,dr = 2\pi i\left(-\frac16(1+i\sqrt3)\right)$$

$$\left(\frac32 - i\frac{\sqrt3}{2}\right)\int_0^\infty \frac{1}{x^3+1}\,dx = \frac{\pi}{3}(\sqrt3 - i)$$

$$\int_0^\infty \frac{1}{x^3+1}\,dx = \frac{2\pi}{3}\,\frac{\sqrt3-i}{3-i\sqrt3} = \frac{2\pi}{3}\,\frac{(\sqrt3-i)(3+i\sqrt3)}{12} = \frac{2\pi}{3}\,\frac{4\sqrt3}{12}$$

$$= \frac{2\pi\sqrt3}{9}\quad\left(=\frac{2\pi}{3\sqrt3}\right)$$

81

1 $f(z)e^{iz} = \dfrac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$, $0<a<b$

(1) $\varphi(z) = \dfrac{e^{iz}}{(z+ia)(z^2+b^2)}$, $\varphi(ia) = \dfrac{e^{-a}}{2ia(-a^2+b^2)}$

(2) $\varphi(z) = \dfrac{e^{iz}}{(z^2+a^2)(z+ib)}$, $\varphi(ib) = \dfrac{e^{-b}}{(-b^2+a^2)2ib}$

$$B_1+B_2 = \frac{1}{2i(a^2-b^2)}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right)$$

$$|f(z)| \le \frac{1}{(R^2-a^2)(R^2-b^2)} \to 0 \quad \text{for} \quad R\to\infty$$

$$\left|\int_{C_R} f(z)e^{iz}\,dz\right| \to 0 \quad \text{for} \quad R\to\infty \quad (\text{jf. Jordans lemma})$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \mathrm{Re}\left(2\pi i(B_1+B_2)\right) = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b}-\frac{e^{-a}}{a}\right)$$

2 $f(z)e^{iaz} = \dfrac{e^{iaz}}{z^2+1}$, $a>0$

$\varphi(z) = \dfrac{e^{iaz}}{z+i}$, $\varphi(i) = \dfrac{e^{-a}}{2i}$, $B = \dfrac{e^{-a}}{2i}$

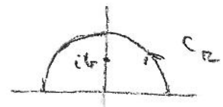
$|f(z)| \le \dfrac{1}{R^2-1} \to 0$ for $R\to\infty$, $\left|\displaystyle\int_{C_R} f(z)e^{iaz}\,dz\right| \to 0$ for $R\to\infty$

fortsættes

2 fortsat

$$\int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \frac{1}{2}\,\mathrm{Re}\left(2\pi i\,\frac{e^{-a}}{2i}\right) = \frac{\pi}{2}\,e^{-a}$$

3 $$f(z)e^{iaz} = \frac{e^{iaz}}{(z^2+b^2)^2}\ , \quad a, b > 0$$



$$\varphi(z) = \frac{e^{iaz}}{(z+ib)^2}\ , \quad \varphi'(z) = \frac{(z+ib)^2\, ia\, e^{iaz} - e^{iaz}\, 2(z+ib)}{(z+ib)^4}$$

$$\varphi'(ib) = \frac{2ib\, ia\, e^{-ab} - 2e^{-ab}}{(2ib)^3} = \frac{(ab+1)\,e^{-ab}}{4ib^3}$$

$$|f(z)| = \frac{1}{R^2-b^2} \to 0 \text{ for } R\to\infty\ , \quad \left|\int_{C_R} f(z)e^{iaz}\,dz\right| \to 0 \text{ for } R\to\infty$$

$$\int_0^\infty \frac{\cos ax}{(x^2+b^2)^2}\,dx = \frac{1}{2}\,\mathrm{Re}\left(2\pi i\,\frac{(ab+1)\,e^{-ab}}{4ib^3}\right) = \frac{\pi\,(ab+1)\,e^{-ab}}{4b^3}$$

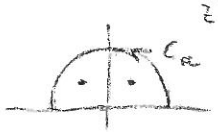
4 $$f(z)e^{iz} = \frac{z\,e^{i2z}}{z^2+3}$$

$$\varphi(z) = \frac{z\,e^{i2z}}{z+i\sqrt{3}}\ , \quad \varphi(i\sqrt{3}) = \frac{i\sqrt{3}\,e^{-2\sqrt{3}}}{2i\sqrt{3}} = \frac{e^{-2\sqrt{3}}}{2} = B$$

$$|f(z)| \le \frac{R}{R^2-3} \to 0 \text{ for } R\to\infty\ , \quad \left|\int f(z)e^{i2z}\,dz\right| \to 0 \text{ for } R\to\infty$$

$$\int_0^\infty \frac{x\sin 2x}{x^2+3}\,dx = \frac{1}{2}\,\mathrm{Im}\left(2\pi i\,\frac{e^{-2\sqrt{3}}}{2}\right) = \frac{\pi}{2}\,e^{-2\sqrt{3}}$$

5 $$f(z)e^{iz} = \frac{z\,e^{iaz}}{z^4+4} = \frac{p(z)}{q(z)}\ , \quad q(z) = 0 \text{ for } z^2 = \pm 2i\ ; \quad a > 0$$



$$q'(z) = 4z^3\ , \quad \frac{p(z)}{q'(z)} = \frac{e^{iaz}}{4z^2}\ , \quad z = \begin{cases} 1+i \\ -1+i \end{cases}$$

$$B_1 + B_2 = \frac{\exp(ia(1+i))}{4\cdot 2i} + \frac{\exp(ia(-1+i))}{4(-2i)}$$

$$= \frac{e^{-a}}{8i}\left(e^{ia} - e^{-ia}\right) = \frac{1}{4}\,e^{-a}\sin a$$

$$|f(z)| \le \frac{R}{R^4-4} \to 0 \text{ for } R\to\infty\ , \quad \left|\int_{C_R} f(z)e^{iz}\,dz\right| \to 0 \text{ for } R\to\infty$$

$$\int_{-\infty}^\infty \frac{x\sin ax}{x^4+4}\,dx = \mathrm{Im}\left(2\pi i\,\frac{1}{4}\,e^{-a}\sin a\right) = \frac{\pi}{2}\,e^{-a}\sin a$$

6 $$f(z)e^{iz} = \frac{z^3\,e^{iaz}}{z^4+4} = \frac{p(z)}{q(z)}\ , \quad a>0\ , \quad \frac{p(z)}{q'(z)} = \frac{1}{4}\,e^{iaz}$$

$$B_1 + B_2 = \frac{1}{4}\,e^{-a}\left(e^{ia} - e^{-ia}\right) = \frac{1}{2}\,e^{-a}\cos a \quad (\text{jvf. 5 udregning})$$

fortsættes

6 fortsat

$$|f(z)| \le \frac{R^3}{R^4-4} \to 0 \text{ for } R\to\infty, \quad \left|\int_{C_R} f(z)e^{iz}\,dz\right| \to 0 \text{ for } R\to\infty$$

$$\int_{-\infty}^{\infty} \frac{x^2 \sin ax}{x^4+4}\,dx = \mathrm{Im}\left(2\pi i\, \tfrac{1}{2}\, e^{-a}\cos a\right) = \pi e^{-a}\cos a$$

9 $$f(z)e^{iz} = \frac{e^{iz}}{z^2+4z+5}, \quad z^2+4z+5=0 \text{ for } z = \frac{-4\pm 2i}{2} = -2\pm i$$

$$\phi(z) = \frac{e^{iz}}{z+2+i}, \quad \phi(-2+i) = \frac{e^{i(-2+i)}}{2i} = \frac{e^{-1-2i}}{2i} = B$$

$$|f(z)| \le \frac{1}{(R-\sqrt{5})^2} \to 0 \text{ for } R\to\infty, \quad \left|\int_{C_R} f(z)e^{iz}\,dz\right| \to 0 \text{ for } R\to\infty$$

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\,dx = \mathrm{Im}\left(2\pi i\, \frac{1}{2i}\, e^{-1}(\cos 2 - i\sin 2)\right) = -\frac{\pi}{e}\sin 2$$

10 $$f(z)e^{iz} = \frac{(z+1)e^{iz}}{z^2+4z+5}, \quad \phi(z) = \frac{(z+1)e^{iz}}{z+2+i}, \quad \phi(-2+i) = \frac{(-1+i)e^{-1-2i}}{2i} = B$$

$$|f(z)| \le \frac{R+1}{(R-\sqrt{5})^2} \to 0 \text{ for } R\to\infty, \quad \left|\int_{C_R} f(z)e^{iz}\,dz\right| \to 0 \text{ for } R\to\infty$$

$$\text{P.V.}\int_{-\infty}^{\infty} \frac{(1+x)\cos x}{x^2+4x+5}\,dx = \mathrm{Re}\left(2\pi i\, \frac{1}{2i}(-1+i)\,\frac{1}{e}(\cos 2 - i\sin 2)\right)$$

$$= \frac{\pi}{e}(\sin 2 - \cos 2) \qquad \text{(opg. 9 udnyttet)}$$

12 a

$$f(z) = e^{iz^2}$$

$$\int_0^R e^{ix^2}\,dx + \int_{C_R} e^{iz^2}\,dz + \int_{L_R} e^{iz^2}\,dz = 0 \qquad \text{(C-G)}$$

$$L_R: \; z = re^{i\frac{\pi}{4}}, \quad dz = e^{i\frac{\pi}{4}}\,dr$$

$$\int_{L_R} e^{iz^2}\,dz = -\int_0^R e^{ir^2 e^{i\frac{\pi}{2}}}\, e^{i\frac{\pi}{4}}\,dr = -\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\int_0^R e^{-r^2}\,dr$$

$$\int_0^R \cos x^2\,dx = \frac{\sqrt{2}}{2}\int_0^R e^{-r^2}\,dr - \mathrm{Re}\int_{C_R} e^{iz^2}\,dz$$

$$\int_0^R \sin x^2\,dx = \frac{\sqrt{2}}{2}\int_0^R e^{-r^2}\,dr - \mathrm{Im}\int_{C_R} e^{iz^2}\,dz$$

b $$C_R: \; z = Re^{i\theta}, \quad dz = Re^{i\theta}\, i\, d\theta$$

$$\int_{C_R} e^{iz^2}\,dz = \int_0^{\frac{\pi}{4}} e^{iR^2 e^{i2\theta}}\, R\, i\, e^{i\theta}\,d\theta, \qquad \begin{array}{l}\varphi = 2\theta \\ d\varphi = 2\,d\theta\end{array}$$

fortsættes

12 b fortsat

$$= \frac{iR}{2} \int_0^{\frac{\pi}{2}} e^{iR^2 e^{i\varphi}} e^{i\frac{\varphi}{2}} \, d\varphi$$

$$= \frac{iR}{2} \int_0^{\frac{\pi}{2}} e^{iR^2(\cos\varphi + i\sin\varphi)} e^{i\frac{\varphi}{2}} \, d\varphi$$

$$= \frac{iR}{2} \int_0^{\frac{\pi}{2}} e^{iR^2\cos\varphi} e^{-R^2\sin\varphi} e^{i\frac{\varphi}{2}} \, d\varphi$$

$$\left| \int_{C_R} e^{iz^2} dz \right| \leq \frac{R}{2} \int_0^{\frac{\pi}{2}} e^{-R^2 \sin\varphi} \, d\varphi \leq \frac{R}{2} \frac{\pi}{2R^2} = \frac{\pi}{4R} \to 0$$

for $R \to \infty$

c

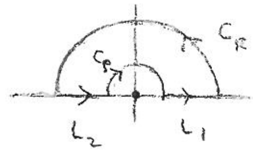
$$\int_0^\infty e^{-x^2} dx = \sqrt{\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy} = \sqrt{\int_0^{\frac{\pi}{2}} \int_0^R e^{-R^2} R \, dR}$$

$$= \sqrt{\frac{\pi}{2} \frac{1}{2} \left[-e^{-R^2} \right]_0^\infty} = \frac{\sqrt{\pi}}{2}$$

$$\frac{\sqrt{2}}{2} \int_0^\infty e^{-x^2} dx = \frac{\sqrt{2\pi}}{4}$$

$$\int_0^\infty \cos x^2 \, dx = \int_0^\infty \sin x^2 \, dx = \frac{\sqrt{2\pi}}{4} \quad \left(= \frac{\sqrt{\pi}}{2\sqrt{2}} \right)$$

1 i $\quad f(z) = \dfrac{e^{iaz} - e^{ibz}}{z^2} \quad , \quad a, b \geq 0$



$z = 0 \; : \quad \varphi(z) = e^{iaz} - e^{ibz}$

$\varphi'(z) = ia e^{iaz} - ib e^{ibz} \quad , \quad \varphi'(0) = i(a-b) = B_0$

$L_1 : \; z = r \, , \; \rho < r < R \, , \qquad L_2 : \; z = -r \, , \; \rho < r < R$

$$\int_{L_1 \cup L_2} f(z)\,dz = \int_\rho^R \left(\frac{\cos ar - \cos br}{r^2} - \frac{\cos a(-r) - \cos b(-r)}{(-r)^2}(-1) \right) dr$$

$$= 2 \int_\rho^R \frac{\cos ar - \cos br}{r^2}\,dr$$

$$\left| \int_{C_R} f(z)\,dz \right| \leq \frac{1+1}{R^2} \pi R \to 0 \quad \text{for } R \to \infty$$

$$\int_{C_\rho} f(z)\,dz \to -i\pi B_0 = \pi(a-b) \quad \text{for } \rho \to 0$$

$$2 \int_0^\infty \frac{\cos ar - \cos br}{r^2}\,dr + 0 + \pi(a-b) = 0$$

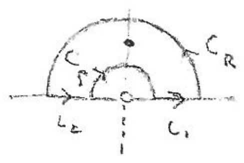
$$\int_0^\infty \frac{\cos ax - \cos bx}{x^2}\,dx = \frac{\pi}{2}(b-a)$$

ii $\quad a = 0 \, , \; b = 2 \; : \quad \displaystyle\int_0^\infty \frac{1 - \cos 2x}{x^2}\,dx = \frac{\pi}{2}(2-0) = \pi$

$$\cos 2x = 1 - 2\sin^2 x \iff 1 - \cos 2x = 2\sin^2 x$$

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$$

2 $\quad f(z) = \dfrac{z^a}{(z^2+1)^2} \quad , \quad -1 < a < 3$



$z = i \; : \quad \varphi(z) = \dfrac{z^a}{(z+i)^2}$

$$\varphi'(z) = \frac{(z+i)^2 a z^{a-1} - z^a 2(z+i)}{(z+i)^4} = \frac{a(z+i)z^{a-1} - 2z^a}{(z+i)^3}$$

$$\varphi'(i) = \frac{a 2i \exp((a-1) i \frac{\pi}{2}) - 2\exp(ai\frac{\pi}{2})}{-8i} = \frac{(1-a)\exp(i\frac{a\pi}{2})}{4i} = B$$

$$\int_{L_1 \cup L_2} f(z)\,dz = \int_\rho^R \left(\frac{r^a}{(r^2+1)^2} - \frac{(-r)^a}{((-r)^2+1)}(-1) \right) dr \, , \quad (-r)^a = r^a(-1)^a = r^a \exp(ai\pi)$$

$$= (1 + \exp(ia\pi)) \int_\rho^R \frac{r^a}{(r^2+1)^2}\,dr$$

fortsættes

2 fortsat

$$\left| \int_{C_R} f(z)\,dz \right| \le \frac{R^a}{(R^2-1)^2}\,\pi R = \frac{\pi R^{a+1}}{(R^2-1)^2} \to 0 \text{ for } R \to \infty, \text{ idet } a+1<4$$

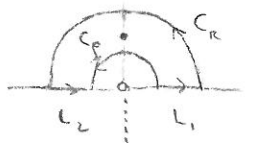
$$\left| \int_{C_\rho} f(z)\,dz \right| \le \frac{\rho^a}{(1-\rho^2)^2}\,\pi \rho = \frac{\pi \rho^{a+1}}{(1-\rho^2)^2} \to 0 \text{ for } \rho \to 0, \text{ idet } a+1>0$$

$$(1+\exp(ia\pi)) \int_0^\infty \frac{r^a}{(r^2+1)^2}\,dr + 0 + 0 = 2\pi i\, B = \frac{\pi}{2}(1-a)\exp\left(i\frac{a\pi}{2}\right)$$

$$\int_0^\infty \frac{x^a}{(x^2+1)^2}\,dx = \frac{\pi(1-a)\exp\left(i\frac{a\pi}{2}\right)}{2(1+\exp(ia\pi))} = \frac{\pi(1-a)}{2\left(\exp\left(-i\frac{a\pi}{2}\right)+\exp\left(i\frac{a\pi}{2}\right)\right)}$$

$$= \frac{(1-a)\pi}{4\cos\left(\frac{a\pi}{2}\right)}$$

4 $\quad f(z) = \dfrac{(\log z)^2}{z^2+1}$



$$z = i: \quad \varphi(z) = \frac{(\log z)^2}{z+i}$$

$$\varphi(i) = \frac{\left(i\frac{\pi}{2}\right)^2}{2i} = -\frac{\pi^2}{8i} = B$$

$$\int_{L_1 \cup L_2} f(z)\,dz = \int_\rho^R \left(\frac{(\ln r)^2}{r^2+1} - \frac{(\ln r + i\pi)^2}{(-r)^2+1}(-1) \right) dr$$

$$= 2\int_\rho^R \frac{(\ln r)^2}{r^2+1}\,dr - \underbrace{\pi^2 \int_\rho^R \frac{1}{r^2+1}\,dr}_{\to\ \pi^2 \frac{\pi}{2} = \frac{\pi^3}{2}} - i\,2\pi \int_\rho^R \frac{\ln r}{r^2+1}\,dr$$

jf. opg. 79 1

$$\left| \int_{C_R} f(z)\,dz \right| \le \frac{(\ln R + \pi)^2}{R^2-1}\,\pi R = \frac{\left(\frac{\ln R}{\sqrt{R}} + \frac{\pi}{\sqrt{R}} \right)^2}{1 - \frac{1}{R^2}} \to 0 \text{ for } R \to \infty$$

$$\left| \int_{C_\rho} f(z)\,dz \right| \le \frac{(-\ln\rho + \pi)^2}{1-\rho^2}\,\pi\rho = \frac{\pi\rho(\ln\rho)^2 - 2\pi^2\rho\ln\rho + \pi^3\rho}{1-\rho^2} \to 0 \text{ for } \rho \to 0,$$

idet
$$\lim_{\rho\to 0} \rho(\ln\rho)^2 = \lim_{\rho\to 0} \frac{(\ln\rho)^2}{\frac{1}{\rho}} = \lim_{\rho\to 0} \frac{2\ln\rho\,\frac{1}{\rho}}{-\frac{1}{\rho^2}} = \lim_{\rho\to 0} \frac{-2\ln\rho}{\frac{1}{\rho}}$$

$$= \lim_{\rho\to 0} \frac{-2\frac{1}{\rho}}{-\frac{1}{\rho^2}} = \lim_{\rho\to 0} 2\rho = 0$$

$$2\int_0^\infty \frac{(\ln r)^2}{r^2+1}\,dr - \frac{\pi^3}{2} + i\,2\pi \int_0^\infty \frac{\ln r}{r^2+1}\,dr + 0 + 0 = 2\pi i\, B$$

$$= -\frac{\pi^3}{4}$$

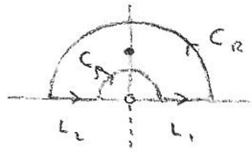
fortsættes

4 fortsat

$$2\int_0^\infty \frac{(\ln x)^2}{x^2+1}\,dx = \frac{\pi^3}{2} - \frac{\pi^3}{4}\,, \qquad \int_0^\infty \frac{(\ln x)^2}{x^2+1}\,dx = \frac{\pi^3}{8}$$

$$2\pi\int_0^\infty \frac{\ln x}{x^2+1}\,dx = 0\,, \qquad \int_0^\infty \frac{\ln x}{x^2+1}\,dx = 0$$

6 a $\quad f(z) = \dfrac{z^{-\frac12}}{z^2+1}$



$$z=i: \quad \varphi(z) = \frac{z^{-\frac12}}{z+i} = \frac{\exp(-\frac12 \log z)}{z+i}$$

$$\varphi(i) = \frac{\exp(-\frac12 i \frac{\pi}{2})}{2i} = \frac{\frac{\sqrt2}{2} - i\frac{\sqrt2}{2}}{2i} = -\frac{\sqrt2}{4} - i\frac{\sqrt2}{4} = B$$

$$\int_{L_1\cup L_2} f(z)\,dz = \int_\rho^R \left(\frac{r^{-\frac12}}{r^2+1} - \frac{(-r)^{-\frac12}}{(-r)^2+1}(-1)\right)dr\,, \qquad \begin{aligned}(-r)^{-\frac12} &= r^{-\frac12}(-1)^{-\frac12}\\ &= r^{-\frac12}\exp(-\tfrac12 i\pi) = -i\,r^{-\frac12}\end{aligned}$$

$$= (1-i)\int_\rho^R \frac{r^{-\frac12}}{r^2+1}\,dr$$

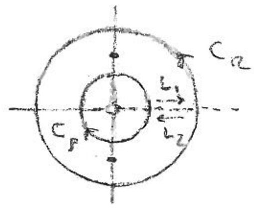
$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{R^{-\frac12}}{R^2-1}\pi R = \frac{\pi R^{\frac12}}{R^2-1} \to 0 \text{ for } R\to\infty$$

$$\left|\int_{C_\rho} f(z)\,dz\right| \le \frac{\rho^{-\frac12}}{1-\rho^2}\pi \rho = \frac{\pi \rho^{\frac12}}{1-\rho^2} \to 0 \text{ for } \rho\to 0$$

$$(1-i)\int_0^\infty \frac{r^{-\frac12}}{r^2+1}\,dr + 0 + 0 = 2\pi i B = \pi\left(\frac{\sqrt2}{2} - i\frac{\sqrt2}{2}\right)$$

$$\int_0^\infty \frac{1}{\sqrt{x}\,(x^2+1)}\,dx = \frac{\pi\sqrt2}{2} \quad \left(= \frac{\pi}{\sqrt2}\right)$$

b $\quad f(z) = \dfrac{z^{-\frac12}}{z^2+1}$



$z=i: \quad B_1 = -\frac{\sqrt2}{4} - i\frac{\sqrt2}{4}$ jf. a

$z=-i: \quad B_2 = \frac{\sqrt2}{4} - i\frac{\sqrt2}{4}$ analogt

$B_1 + B_2 = -i\frac{\sqrt2}{2}$

$$\int_{L_1\cup L_2} f(z)\,dz = \int_\rho^R \left(\frac{r^{-\frac12}}{r^2+1} - \frac{r^{-\frac12}\exp(-\frac12 i 2\pi)}{r^2+1}\right)dr = 2\int_\rho^R \frac{r^{-\frac12}}{r^2+1}\,dr$$

$$2\int_0^\infty \frac{r^{-\frac12}}{r^2+1}\,dr + \underbrace{0+0}_{\text{jf. a}} = 2\pi i\,(B_1+B_2) = \pi\sqrt2$$

$$\int_0^\infty \frac{1}{\sqrt{x}\,(x^2+1)}\,dx = \frac{\pi\sqrt2}{2} \quad \left(= \frac{\pi}{\sqrt2}\right)$$

1 $$\int_0^{2\pi} \frac{1}{5+4\sin\theta}\,d\theta \;,\quad \sin\theta = \frac{z-z^{-1}}{2i} \;,\quad d\theta = \frac{dz}{iz} \;,\quad |z|\le 1$$

$$f(z) = \frac{\frac{1}{iz}}{5+4\,\frac{z-z^{-1}}{2i}} = \frac{1}{5iz+2z^2+2} = \frac{1}{2z^2+5iz+2}$$

$$2z^2+5iz+2 = 0 \quad \text{for} \quad z = \frac{-5i\pm 3i}{4} = \begin{cases} -\frac{1}{2}i \\ -2i \end{cases}$$

$$f(z) = \frac{1}{2\,(z+\frac{1}{2}i)(z+2i)}$$

$$z = -\tfrac{1}{2}i : \quad \varphi(z) = \frac{1}{2(z+2i)} \;,\qquad \varphi(-\tfrac{1}{2}i) = \frac{1}{3i} = B$$

$$\int_0^{2\pi} \frac{1}{5+4\sin\theta}\,d\theta = 2\pi i\,B = \frac{2\pi}{3}$$

3 $$\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\,d\theta \;,\quad \cos 3\theta = \frac{z^3+z^{-3}}{2} \;,\quad \cos 2\theta = \frac{z^2+z^{-2}}{2} \;,$$

$$d\theta = \frac{dz}{iz} \;,\quad |z|\le 1$$

$$f(z) = \frac{\left(\frac{z^3+z^{-3}}{2}\right)^2 \frac{1}{iz}}{5-4\,\frac{z^2+z^{-2}}{2}} = \frac{z^2(z^6+1)^2}{4z^6\,iz\,(5z^2-2z^4-2)}$$

(1) $$f(z) = -\frac{1}{4i}\,\frac{z^{12}+2z^6+1}{2z^9-5z^7+2z^5} = -\frac{1}{4i}\left(\frac{1}{2}z^3+\frac{5}{4}z+\frac{21}{8}\frac{1}{z}+\frac{101}{16}\frac{1}{z^3}+\dots\right)$$

$$B_0 = -\frac{21}{32i}$$

(2) $$2z^4-5z^2+2 = 0 \quad \text{for} \quad z^2 = \frac{5\pm 3}{4} = \begin{cases} 2 \\ \frac{1}{2} \end{cases} \;,\quad z^2 = \tfrac{1}{2} \;\text{for}\; z = \pm\frac{\sqrt{2}}{2}$$

$$z = \pm\frac{\sqrt{2}}{2} : \quad \varphi(z) = \frac{(z^6+1)^2}{-4i\,z^5\,2\,(z^2-2)(z\pm\frac{\sqrt{2}}{2})}$$

$$\varphi(\pm\tfrac{\sqrt{2}}{2}) = \frac{(\frac{1}{8}+1)^2}{-4i\,\frac{1}{4}\,(\pm\frac{\sqrt{2}}{2})\,2\,(-\frac{3}{2})(\pm\sqrt{2})} = \frac{27}{64i}$$

$$B_1 = B_2 = \frac{27}{64i} \;,\qquad B_1+B_2 = \frac{27}{32i}$$

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\,d\theta = 2\pi i\,(B_0+B_1+B_2) = 2\pi i\left(-\frac{21}{32i}+\frac{27}{32i}\right) = \frac{3\pi}{8}$$

5 $$\int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta \;,\quad \cos\theta = \frac{z+z^{-1}}{2} \;,\quad \cos 2\theta = \frac{z^2+z^{-2}}{2} \;,$$

$$d\theta = \frac{dz}{iz} \;,\quad |z|\le 1 \;,\quad |a|<1$$

fortsattes

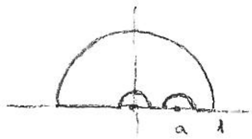
85

5 fortsat

$$f(z) = \frac{\frac{z^2+z^{-2}}{2}\,\frac{1}{iz}}{1+2a\frac{z+z^{-1}}{2}+a^2} = \frac{z^4+1}{2iz^2\left((1+a^2)z-az^2-a\right)}$$

$$az^2-(1+a^2)z+a=0 \quad \text{for} \quad z=\frac{1-a^2\pm(1-a^2)}{2a} = \begin{cases}\frac{1}{a}\\ a\end{cases}$$

$$f(z) = -\frac{1}{2i}\,\frac{z^4+1}{az^2(z-a)(z-\frac{1}{a})}$$



$z=0$: $$q(z) = -\frac{1}{2i}\,\frac{z^4+1}{az^2-(1+a^2)z+a}$$

$$q'(z) = -\frac{1}{2i}\,\frac{(az^2-(1+a^2)z+a)4z^3-(z^4+1)(2az-(1+a^2))}{((az^2-(1+a^2)z+a)^2}$$

$$q'(0) = -\frac{1}{2i}\,\frac{1+a^2}{a^2} = B_0$$

$z=a$: $$q(z) = -\frac{1}{2i}\,\frac{z^4+1}{z^2(az-1)}, \quad q(a) = -\frac{1}{2i}\,\frac{a^4+1}{a^2(a^2-1)} = B_1$$

$$B_0+B_1 = -\frac{1}{2i}\,\frac{1}{a^2}\left(1+a^2+\frac{a^4+1}{a^2-1}\right) = -\frac{1}{2i}\,\frac{1}{a^2}\,\frac{2a^4}{a^2-1} = \frac{1}{i}\,\frac{a^2}{1-a^2}$$

$$\int_0^\pi \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta = \pi i(B_0+B_1) = \frac{\pi a^2}{1-a^2}$$

7 $\int_0^\pi \sin^{2n}\theta\,d\theta$, $n=1,2,\dots$, $\sin\theta = \frac{z+z^{-1}}{2i}$, $d\theta = \frac{dz}{iz}$, $|z|\le 1$

$$f(z) = \left(\frac{z-z^{-1}}{2i}\right)^{2n}\frac{1}{iz} = \frac{(-1)^n}{i2^{2n}}\,\frac{1}{z}\,(z-z^{-1})^{2n}$$

$$= \frac{(-1)^n}{i2^{2n}}\,\frac{1}{z}\sum_{k=0}^{2n}\binom{2n}{k}z^k(-z^{-1})^{2n-k}$$

$$= \frac{1}{i2^{2n}}\,\frac{1}{z}\sum_{k=0}^{2n}\binom{2n}{k}(-1)^{n-k}z^{2(k-n)}$$

$$B_0 = \frac{1}{i2^{2n}}\binom{2n}{n} = \frac{1}{i2^{2n}}\,\frac{(2n)!}{(n!)^2}$$

$$\int_0^\pi \sin^{2n}\theta\,d\theta = \pi i B_0 = \frac{(2n)!}{2^{2n}(n!)^2}\,\pi$$

87

1 $C: |z|=1$

a $f(z)=z^2$, $Z=2$, $P=0$, $\Delta_C \arg f(z) = (2-0)2\pi = 4\pi$

b $f(z) = \frac{z^3+2}{z}$, $Z=0$, $P=1$, $\Delta_C \arg f(z) = (0-1)2\pi = -2\pi$

c $f(z) = \frac{(2z-1)^7}{z^3}$, $Z=7$, $P=3$, $\Delta_C \arg f(z) = (7-3)2\pi = 8\pi$

6 $\quad C: \ |z| = 1$

a $\quad P(z) = z^6 - 5z^4 + z^3 - 2z$

$f(z) = -5z^4 \qquad |f(z)| = 5$ på C

$g(z) = z^6 + z^3 - 2z \qquad |g(z)| \le 1 + 1 + 2 = 4$ på C

$P(z)$ har 4 rødder inden for C.

b $\quad P(z) = 2z^4 - 2z^3 + 2z^2 - 2z + 9$

$f(z) = 9 \qquad |f(z)| = 9$ på C

$g(z) = 2z^4 - 2z^3 + 2z^2 - 2z \qquad |g(z)| \le 2 + 2 + 2 + 2 = 8$ på C

$P(z)$ har ingen rødder inden for C.

8 $\quad P(z) = 2z^5 - 6z^2 + z + 1$

$|z| = 2: \quad f(z) = 2z^5 \qquad |f(z)| = 64$

$g(z) = -6z^2 + z + 1 \qquad |g(z)| \le 24 + 2 + 1 = 29$

$P(z)$ har 5 rødder inden for $|z| = 2$

$|z| = 1: \quad f(z) = -6z^2 \qquad |f(z)| = 6$

$g(z) = 2z^5 + z + 1 \qquad |g(z)| \le 2 + 2 + 1 = 4$

$P(z)$ har 2 rødder inden for $|z| = 1$

$P(z)$ har $5 - 2 = 3$ rødder i $1 \le |z| < 2$

9 $\quad cz^n = e^z, \quad |c| > e, \quad C: |z| = 1$

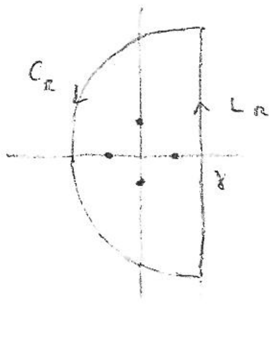
$$C: \left.\begin{array}{l} |cz^n| = |c| \\ |-e^z| = e \end{array}\right\} \quad |cz^n| > |-e^z|$$

$cz^n - e^z$ har samme antal rødder som cz^n inden for C

$cz^n = e^z$ har n løsninger inden for C

89

1 $\quad F(s) = \dfrac{2s^3}{s^4 - 4} = \dfrac{2s^3}{(s^2+2)(s^2-2)}$



sing. pkt'er: $\quad s = \pm i\sqrt{2}, \quad s = \pm\sqrt{2}$

1 fortsat

$$s = \sqrt{2} : \quad \varphi(s) = \frac{2s^3}{(s^2+2)(s+\sqrt{2})}, \quad \varphi(\sqrt{2}) = \frac{2\cdot 2\sqrt{2}}{4\cdot 2\sqrt{2}} = \frac{1}{2}$$

$$s = -\sqrt{2} : \quad \varphi(s) = \frac{2s^3}{(s^2+2)(s-\sqrt{2})}, \quad \varphi(-\sqrt{2}) = \frac{2(-2\sqrt{2})}{4(-2\sqrt{2})} = \frac{1}{2}$$

$$s = i\sqrt{2} : \quad \varphi(s) = \frac{2s^3}{(s+i\sqrt{2})(s^2-2)}, \quad \varphi(i\sqrt{2}) = \frac{2(-2i\sqrt{2})}{2i\sqrt{2}(-4)} = \frac{1}{2}$$

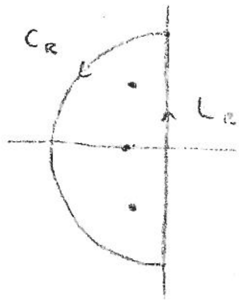
$$s = -i\sqrt{2} : \quad \varphi(s) = \frac{2s^3}{(s-i\sqrt{2})(s^2-2)}, \quad \varphi(-i\sqrt{2}) = \frac{2\cdot 2i\sqrt{2}}{-2i\sqrt{2}(-4)} = \frac{1}{2}$$

$$|F(s)| \le \frac{2(R+\gamma)^3}{(R-\gamma)^4 - 1} \to 0 \quad \text{for } R \to \infty$$

$$\lim_{R\to\infty} \int_{C_R} e^{st} F(s)\, ds = 0$$

$$f(t) = e^{\sqrt{2}t}\tfrac{1}{2} + e^{-\sqrt{2}t}\tfrac{1}{2} + e^{i\sqrt{2}t}\tfrac{1}{2} + e^{-i\sqrt{2}t}\tfrac{1}{2}$$
$$= \cosh\sqrt{2}t + \cos\sqrt{2}t$$

2 $$F(s) = \frac{2s-2}{(s+1)(s^2+2s+5)}$$



$\overline{F(s)} = F(\bar{s})$

sing. pkt'er: $s=-1$, $s = -1 \pm 2i$

$$s = -1 : \quad \varphi(s) = \frac{2s-2}{s^2+2s+5}, \quad \varphi(-1) = -1$$

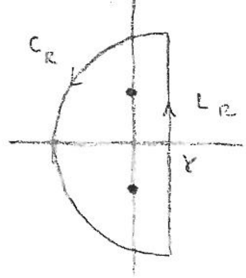
$$s = -1+2i : \quad \varphi(s) = \frac{2s-2}{(s+1)(s+1+2i)}, \quad \varphi(-1+2i) = \frac{-2+4i-2}{2i\cdot 4i} = \frac{1}{2} - \frac{1}{2}i$$

$$|F(s)| \le \frac{2R-2}{(R-1)(R-\sqrt{5})^2} \to 0 \quad \text{for } R \to \infty$$

$$\lim_{R\to\infty} \int_{C_R} e^{st} F(s)\, ds = 0$$

$$f(t) = e^{-t}(-1) + 2e^{-t}\,\mathrm{Re}\left[e^{i2t}\left(\tfrac{1}{2} - \tfrac{1}{2}i\right)\right]$$
$$= -e^{-t} + 2e^{-t}\left(\tfrac{1}{2}\cos 2t + \tfrac{1}{2}\sin 2t\right)$$
$$= e^{-t}(\cos 2t + \sin 2t - 1)$$

5 $$F(s) = \frac{8a^3 s^2}{(s^2+a^2)^3}, \quad a > 0, \quad \overline{F(s)} = F(\bar{s})$$



sing. pkt'er: $s = \pm ia$, poler af 3. orden

fortsættes

5 fortsat

$z = ia$: Principaldelen af $F(s)$, jf. opg. 72 5 :

$$-\frac{i}{2(z-ai)} - \frac{a}{2(z-ai)^2} - \frac{a^2 i}{(z-ai)^3},$$

dvs. $b_1 = -\frac{i}{2}$, $b_2 = -\frac{a}{2}$, $b_3 = -a^2 i$

$$|F(s)| \le \frac{8a^3(R+\gamma)^3}{((R-\gamma)^2-a^2)^3} \to 0 \text{ for } R \to \infty$$

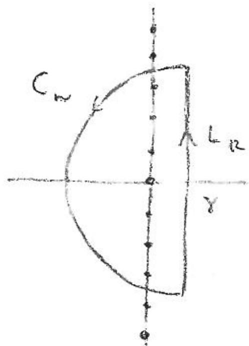
$$\lim_{R\to\infty} \int_{C_R} e^{st} F(s)\, ds = 0$$

$$f(t) = 2e^{0t}\, \mathrm{Re}\left[e^{iat}\left(-\frac{i}{2} - \frac{a}{2}t - \frac{a^2 i}{2!}t^2 \right)\right]$$

$$= 2\left(\tfrac{1}{2}\sin 2t - \tfrac{1}{2}at\cos 2t + \tfrac{1}{2}a^2t^2 \sin 2t \right)$$

$$= (1+a^2t^2)\sin 2t - at\cos 2t$$

10 $F(s) = \frac{1}{s^2} - \frac{1}{s\sinh s}$, $\overline{F(s)} = F(\bar{s})$



sing. punkter : $s = 0$ (hævelig sing.)

$s = \pm n\pi i$, $n \in \mathbb{Z}_+$, simple poler

$s = 0$:
$$\frac{1}{s^2} - \frac{1}{s\sinh s} = \frac{1}{s^2} - \left(\frac{1}{s^2} - \frac{1}{6} + \frac{7}{360}s^2 + \dots \right) \quad \text{jf. 67 eks 2}$$
$$= \frac{1}{6} - \frac{7}{360}s^2 - \dots), \quad B_0 = 0$$

$s = n\pi$:
$$\frac{1}{s\sinh s} = \frac{p(s)}{q(s)}, \quad q'(s) = \sinh s + s\cosh s$$
$$q'(n\pi i) = 0 + n\pi i(-1)^n$$

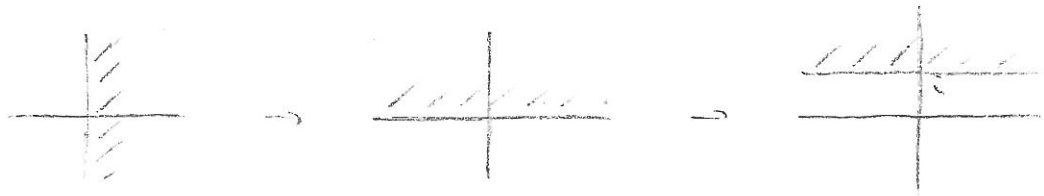
$$B_n = \frac{(-1)^n}{n\pi i}$$

$$f(t) = -\sum_{n=1}^{\infty} 2e^{0t}\, \mathrm{Re}\left[e^{in\pi t}\frac{(-1)^n}{n\pi i} \right] \quad \text{(formelt)}$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin n\pi t$$

90

2 $\quad w = iz + i \quad , \quad Z = iz. \quad w = Z + i$

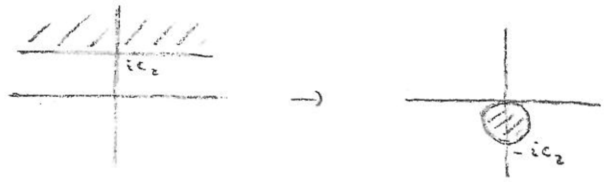


4 $\quad w = (1-i)z = \sqrt{2}\, e^{i(-\frac{\pi}{4})} r e^{i\theta} = \sqrt{2}\, r e^{i(-\frac{\pi}{4}+\theta)}$

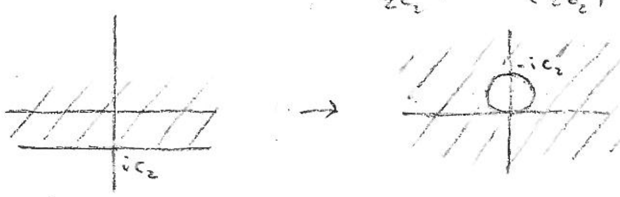
92

3 $\quad w = \frac{1}{z} = \frac{x}{x^2+y^2} + i\,\frac{-y}{x^2+y^2} \quad , \quad z = \frac{1}{w} = \frac{u}{u^2+v^2} + i\,\frac{-v}{u^2+v^2}$

$y > c_2 > 0 : \quad \frac{-v}{u^2+v^2} > c_2 \iff u^2+v^2+\frac{v}{c_2} < 0 \iff u^2 + \left(v + \frac{1}{2c_2}\right)^2 < \left(\frac{1}{2c_2}\right)^2$

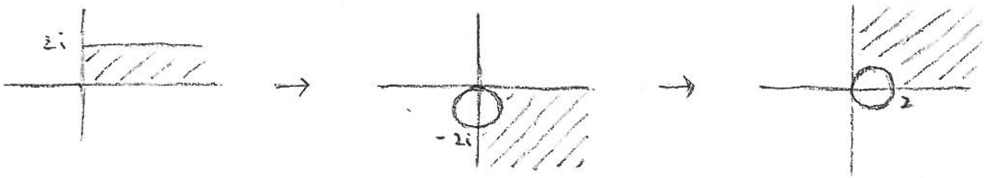


$y > c_2\,,\, c_2 < 0 :$ analogt $\quad u^2 + \left(v + \frac{1}{2c_2}\right)^2 > \left(\frac{1}{2c_2}\right)^2$



$y > 0 :$

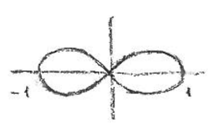
9 $\quad w = \frac{i}{z} \, ; \quad Z = \frac{1}{z} \quad , \quad w = iZ$



10 $\quad x^2 - y^2 = 1 \iff z^2 + \bar{z}^2 = 2$, jf. opg 14 5

$w = \frac{1}{z} \iff z = \frac{1}{w} \quad , \quad w = \rho e^{i\varphi}$

$x^2 - y^2 = 1 \iff \frac{1}{w^2} + \frac{1}{\bar{w}^2} = 2 \iff \frac{1}{\rho^2 e^{i2\varphi}} + \frac{1}{\rho^2 e^{-i2\varphi}} = 2$

$\iff \rho^2 = \frac{e^{i2\varphi} + e^{-i2\varphi}}{2} \iff \rho^2 = \cos 2\varphi$, lemniskat 

92

13 $w = z + \frac{1}{z}$, $z = re^{i\theta}$

i $r = r_0 > 1$

$w_0 = r_0e^{i\theta} + \frac{1}{r_0e^{i\theta}} = r_0(\cos\theta + i\sin\theta) + \frac{1}{r_0}(\cos\theta - i\sin\theta)$

$= (r_0 + \frac{1}{r_0})\cos\theta + i(r_0 - \frac{1}{r_0})\sin\theta$

$$\left.\begin{aligned} u &= (r_0 + \tfrac{1}{r_0})\cos\theta \\ v &= (r_0 - \tfrac{1}{r_0})\sin\theta \end{aligned}\right\}$$ parameterfremstilling for

ellipse med $\begin{cases} \text{storakse } a = r_0 + \frac{1}{r_0} \\ \text{lilleakse } b = r_0 - \frac{1}{r_0} \end{cases}$

brændpunkter: $\pm ae = \pm a\sqrt{1 - \frac{b^2}{a^2}} = \pm\sqrt{a^2 - b^2} = \pm 2$

ii $r_0 = 1$, $u = 2\cos\theta$, $v = 0$

$r \le 1$: $-2 \le u \le 2$

$r > 1$: $\mathbb{C} \setminus \{u \mid -2 \le u \le 2\}$, den komplekse plan — bortset fra liniestykket $-2 \le u \le 2$ — 'udfylder' af ellipserne.

94

1 $w = \frac{az+b}{cz+d}$,

z	w
2	1
i	i
-2	-1

, $1 = \frac{2a+b}{2c+d}$ (1)

$i = \frac{ia+b}{ic+d}$ (2)

$-1 = \frac{-2a+b}{-2c+d}$ (3)

$$\left.\begin{aligned} (1)\quad & 2c+d = 2a+b \\ (3)\quad & 2c-d = -2a+b \end{aligned}\right\} \Rightarrow 4c = 2b \Rightarrow b = 2c \Rightarrow d = 2c$$

(2) $-c + i2c = ia + 2c \Rightarrow ia = 3c \Rightarrow a = -3ic$

$$w = \frac{-3icz + 2c}{cz - 6ic} = \frac{3z + 2i}{iz + 6}$$

2 $w = \frac{az+b}{cz+d}$,

z	w
$-i$	-1
0	i
i	1

, $-1 = \frac{-ia+b}{-ic+d}$ (1)

$i = \frac{b}{d}$ (2)

$1 = \frac{ia+b}{ic+d}$ (3)

(2) $b = id$

$$\left.\begin{aligned} (1)\quad & ic - d = -ia + id \\ (3)\quad & ic + d = ia + id \end{aligned}\right\} \Rightarrow 2ic = 2id \Rightarrow c = d \Rightarrow d = ia$$

$\Rightarrow a = -id$

$$w = \frac{-idz + id}{dz + d} = \frac{-iz + i}{z + 1} = i\,\frac{-z+1}{z+1}$$

fortsættes

94

2 fortsat

$$x = 0: \quad w = u + iv = i\,\frac{-iy+1}{iy+1} = \frac{y+i}{1+iy} = \frac{(y+i)(1-iy)}{1+y^2}$$

$$= \frac{y - iy^2 + i + y}{1+y^2} = \frac{2y}{1+y^2} + i\,\frac{1-y^2}{1+y^2}$$

$$u^2 + v^2 = \frac{4y^2 + (1-y^2)^2}{(1+y^2)^2} = \frac{(1+y^2)^2}{(1+y^2)^2} = 1$$

4 $$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)},$$

z	w
z_1	0
z_2	1
z_3	∞

$$\frac{w-0}{1-0} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$w = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

7 a $$w = \frac{z-1}{z+1}$$

$$\frac{z-1}{z+1} = z \iff z - 1 = z^2 + z \iff z^2 = -1 \iff z = \pm i$$

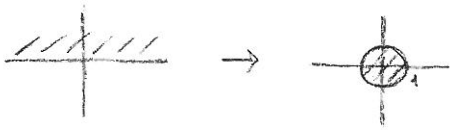
b $$w = \frac{6z-9}{z}$$

$$\frac{6z-9}{z} = z \iff 6z - 9 = z^2 \iff z^2 - 6z + 9 = 0$$

$$\iff (z-3)^2 = 0 \iff z = 3$$

95

1 $$w = \frac{i-z}{i+z},$$



jf. 95 eks. 1

$$y = 0: \quad w = u + iv = \frac{i-x}{i+x} = \frac{(i-x)^2}{-1-x^2} = \frac{-1-i2x+x^2}{-(1+x^2)}$$

$$= \frac{1-x^2}{1+x^2} + i\,\frac{2x}{1+x^2}$$

3 a $$w = \frac{i-z}{i+z} \iff iw + wz = i - z \iff (w+1)z = i(1-w)$$

$$\iff z = i\,\frac{1-w}{1+w} \quad *$$

$$w = i\,\frac{1-z}{1+z},$$

, jf. * og opg. 1

95

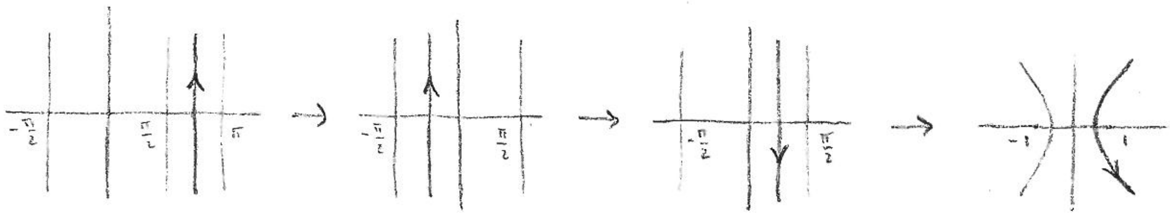
3 b $w = \frac{z-2}{z} = \frac{(z-1)-1}{(z-1)+1} = \frac{Z-1}{Z+1} = -\frac{1-Z}{1+Z} = i\cdot i\,\frac{1-Z}{1+Z} = iW$

$Z = z-1$, $W = i\,\frac{1-Z}{1+Z}$ $\quad w = iW$

96

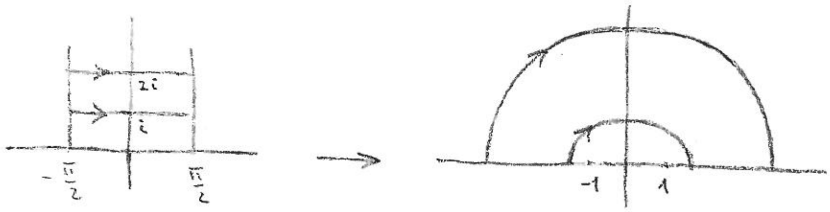
2 $w = \sin z = \sin(\pi - z) = \sin(-(z-\pi))$

$Z = z-\pi$, $W = -Z$, $w = \sin W$



3 $w = \sin z$

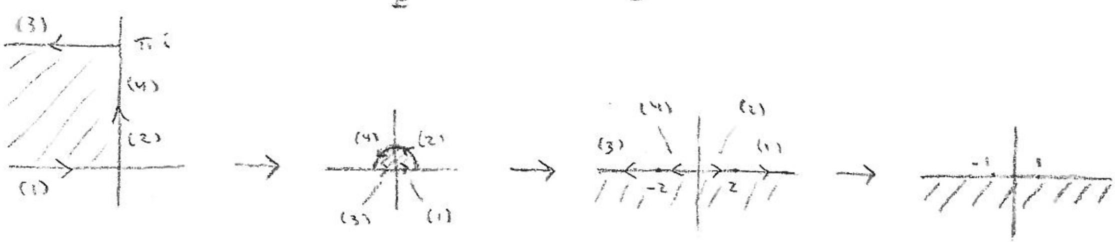
$y = c_2 > 0$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$: $\frac{u^2}{\cosh^2 c_2} + \frac{v^2}{\sinh^2 c_2} = 1$, $v > 0$



halvellipserne 'udfylder' halvplanen $v > 0$

7 $w = \cosh z = \frac{e^z + e^{-z}}{2} = \frac{1}{2}\left(e^z + \frac{1}{e^z}\right)$

$Z = e^z$, $W = Z + \frac{1}{Z}$, $w = \frac{1}{2}W$

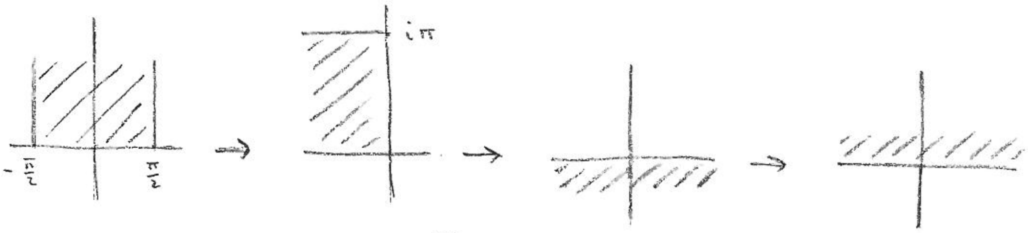


96

8 a $w = \sin z = -\cos\left(z + \frac{\pi}{2}\right) = -\cosh i\left(z + \frac{\pi}{2}\right)$

$Z = i\left(z + \frac{\pi}{2}\right)$, $W = \cosh Z$, $w = -W$

b



jf. opg. 7

97

1 $w = z^2$, $u = x^2 - y^2$, $v = 2xy$

$y = y_1$: $x = \dfrac{v}{2y_1}$, $u = \dfrac{v^2}{4y_1^2} - y_1^2$

$$v^2 = 4y_1^2\,(u + y_1^2)$$

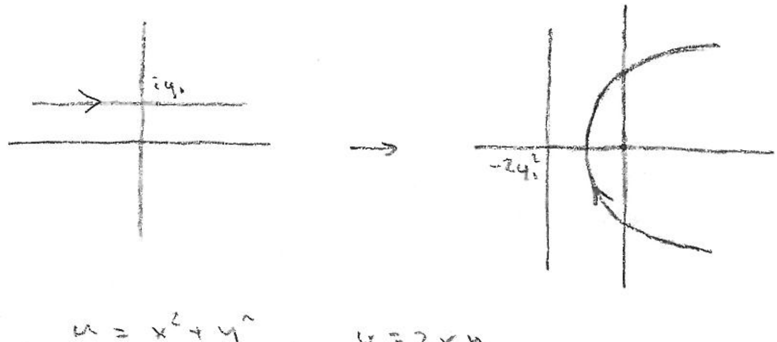
parabel med

parameter $4y_1^2$

toppunkt $(u,v) = (-y_1^2, 0)$

brændpunkt $(u,v) = (0,0)$

ledelinie $u = -2y_1^2$

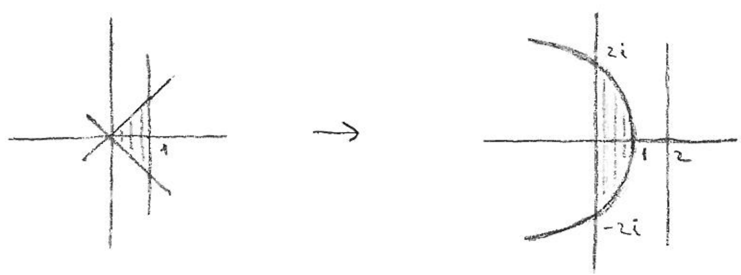


4 $w = z^2$, $u = x^2 + y^2$, $v = 2xy$

$x = x_1$: $v^2 = -4x_1^2\,(u - x_1^2)$, jf. 97 eks 1

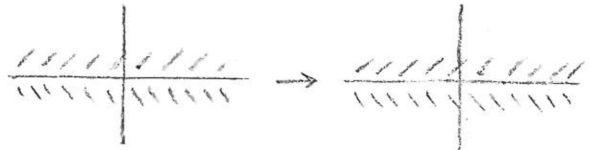
$x_1 = 1$: $v^2 = -4\,(y - 1)$

$y = \pm x$: $u = 0$, $v = \pm 2x^2$; $0 \le x \le 1$: $-2 \le v \le 2$



$\underline{97}$

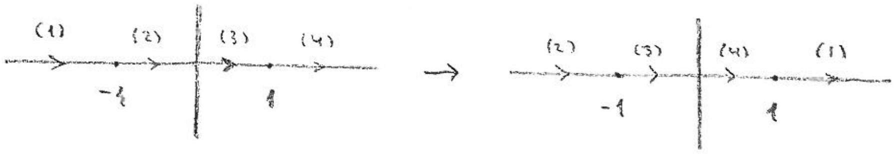
7 $\quad Z = \dfrac{z-1}{z+1}$:



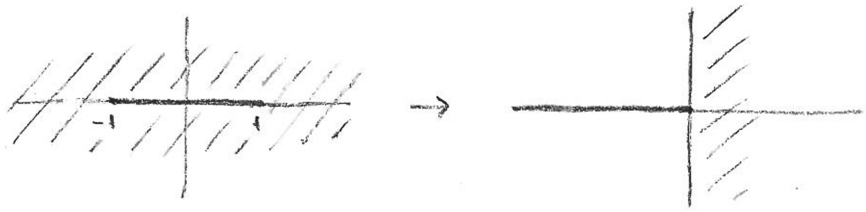
jf. $\underline{95}$ opg. 2

$y = 0$: $Z = \dfrac{x-1}{x+1}$,

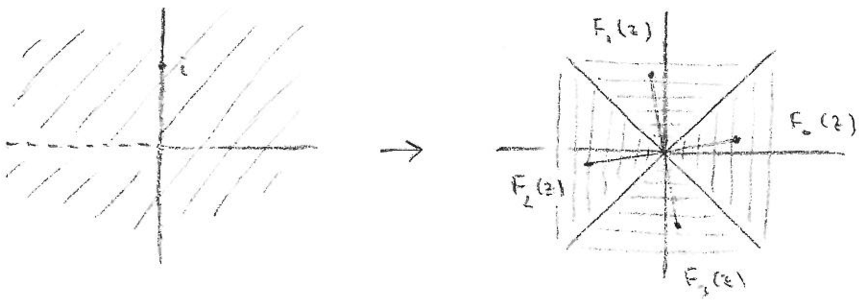
x	$-\infty$	-1^-	-1^+	0	1	∞
Z	1	∞	$-\infty$	-1	0	1



$$\left(\frac{Z-1}{Z+1}\right)^{\frac{1}{2}} = Z^{\frac{1}{2}} :$$



8 $\quad F_k(z) = \sqrt[4]{r}\ \exp i\,\dfrac{\theta + k\,2\pi}{4}$, $\quad r > 0$, $-\pi < \theta < \pi$, $k = 0, 1, 2, 3$



$$F_0(i) = \exp i\,\frac{\pi}{8} = \cos\frac{\pi}{8} + i\sin\frac{\pi}{8} = \sqrt{\frac{1+\cos\frac{\pi}{4}}{2}} + i\sqrt{\frac{1-\cos\frac{\pi}{4}}{2}}$$

$$F_0(i) = \tfrac{1}{2}\sqrt{2+\sqrt{2}} + i\,\tfrac{1}{2}\sqrt{2-\sqrt{2}}$$

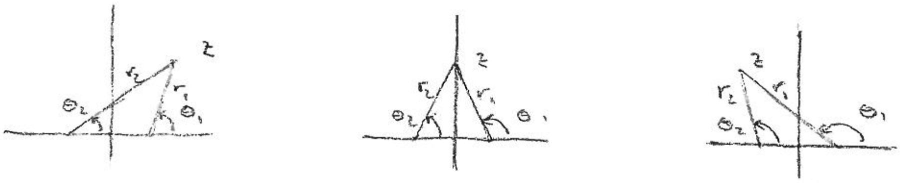
$$F_1(i) = -\tfrac{1}{2}\sqrt{2-\sqrt{2}} + i\,\tfrac{1}{2}\sqrt{2+\sqrt{2}}$$

$$F_2(i) = -\tfrac{1}{2}\sqrt{2+\sqrt{2}} - i\,\tfrac{1}{2}\sqrt{2-\sqrt{2}}$$

$$F_3(i) = \tfrac{1}{2}\sqrt{2-\sqrt{2}} - i\,\tfrac{1}{2}\sqrt{2+\sqrt{2}}$$

$\underline{98}$

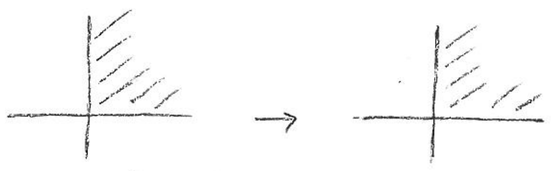
1 $\quad w = (z^2-1)^{\frac{1}{2}} = \sqrt{r_1 r_2}\ \exp i\,\dfrac{\theta_1+\theta_2}{2}$, $\quad r_1, r_2 > 0$, $\quad 0 < \theta_1 + \theta_2 < \pi$



$0 < \dfrac{\theta_1+\theta_2}{2} < \dfrac{\pi}{2}$ $\qquad$ $\dfrac{\theta_1+\theta_2}{2} = \dfrac{\pi}{2}$ $\qquad$ $\dfrac{\pi}{2} < \dfrac{\theta_1+\theta_2}{2} < \pi$

fortsættes

1 fortsat



2 (opgave 1 fortsat)

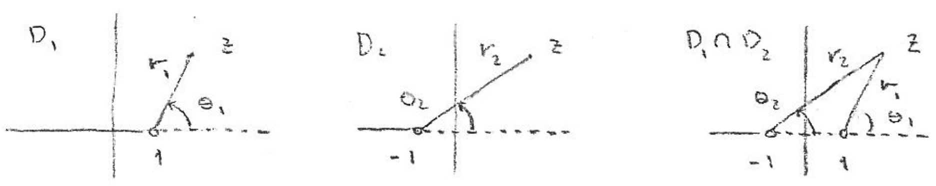
$$w^2 = x^2 - y^2 - 1 + i\,2xy$$

i
$$(r_1 r_2)^2 = |z^2-1|^2 = (x^2-y^2-1)^2 + (2xy)^2$$
$$= x^4 + y^4 + 1 - 2x^2y^2 - 2x^2 + 2y^2 + 4x^2y^2$$
$$= x^4 + y^4 + 1 + 2x^2y^2 + 2x^2 + 2y^2 - 4x^2$$
$$= (x^2+y^2+1)^2 - 4x^2$$
$$r_1 r_2 = \sqrt{(x^2+y^2+1)^2 - 4x^2}$$
$$w = \sqrt{\frac{r_1r_2 + x^2 - y^2 - 1}{2}} + i\sqrt{\frac{r_1r_2 - (x^2-y^2-1)}{2}} \,, \quad \text{idet } x, y > 0$$
$$= \frac{1}{\sqrt{2}}\sqrt{r_1r_2 + x^2 - y^2 - 1} + i\,\frac{1}{\sqrt{2}}\sqrt{r_1r_2 - x^2 + y^2 + 1}$$

ii $x^2 - y^2 = 1 \,, \quad x, y > 0$:
$$w = \frac{1}{\sqrt{2}}\sqrt{r_1r_2 + x^2 - (x^2-1) - 1} + i\,\frac{1}{\sqrt{2}}\sqrt{r_1r_2 - x^2 + (x^2-1) + 1}$$
$$= \frac{1}{\sqrt{2}}\sqrt{r_1r_2} + i\,\frac{1}{\sqrt{2}}\sqrt{r_1r_2} = u + iv \implies u = v$$

6 i
$$w = F(z) = \left(\frac{z-1}{z+1}\right)^{\frac{1}{2}} = \sqrt{\frac{r_1}{r_2}}\,\exp i\,\frac{\theta_1-\theta_2}{2} \,, \qquad \begin{array}{l} r_1, r_2 > 0 \\ 0 < \theta_1 < 2\pi \\ 0 < \theta_2 < 2\pi \end{array}$$

Forgreningssnit:



Betragt også
$$G(z) = \sqrt{\frac{r_1}{r_2}}\,\exp i\,\frac{\theta_1+\theta_2}{2} \,, \qquad \begin{array}{l} r_1, r_2 > 0 \\ -\pi < \theta_1 < \pi \\ -\pi < \theta_2 < \pi \end{array}$$

fortsættes

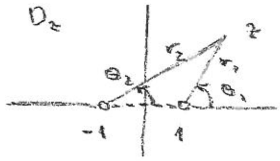
6 i fortsat

G er analytisk i et område, der indeholder $\{x \mid x>1\}$.

F og G stemmer overens i dette område bortset fra punkter på den reelle akse.

Udvidelse af F : Sæt $F(z) = G(z)$ på $\{x \mid x>1\}$

F er nu analytisk i D_z



(Ved passage af $\{x \mid -1<x<1\}$ skifter $F(z)$ fortegn.)

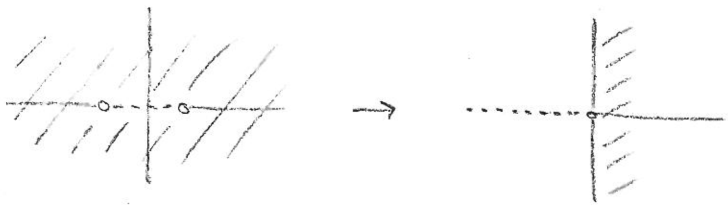
$F(z)$ er entydig, idet

$$F(z_1) = F(z_2) \Rightarrow \left(\frac{z_1-1}{z_1+1}\right)^{\frac{1}{2}} = \left(\frac{z_2-1}{z_2+1}\right)^{\frac{1}{2}} \Rightarrow \frac{z_1-1}{z_1+1} = \frac{z_2-1}{z_2+1}$$

$$\Rightarrow z_1 z_2 + z_1 - z_2 - 1 = z_1 z_2 - z_1 + z_0 - 1$$

$$\Rightarrow 2z_1 = 2z_2 \Rightarrow z_1 = z_2$$

ii



sml. m. opg. 97 7

iii

$$w^2 = \frac{z-1}{z+1}, \quad w^2 z + w^2 = z - 1, \quad (1-w^2)z = 1 + w^2$$

$$z = \frac{1-w^2}{1+w^2}, \quad \text{Re}\, w > 0,$$

$F(z)$ afbilder D_z på $\{w \mid \text{Re}\, w > 0\}$.

7 (opgave 6 fortsat)

$$\frac{z-1}{z+1} = \frac{x-1+iy}{x+1+iy} = \frac{(x-1+iy)(x+1-iy)}{(x+1)^2+y^2} = \frac{x^2-1+y^2+i2xy}{(x+1)^2+y^2}$$

$$= \frac{x^2+y^2-1}{(x+1)^2+y^2} + i\frac{2xy}{(x+1)^2+y^2}$$

$x=0$: se opg. 97 7

$x^2+y^2=1,\ y>0$: $u=0,\ v>0$

fortsættes

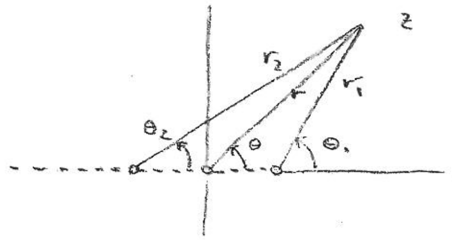
98

7 fortsat

8 $w = F(z) = (z(z^2-1))^{\frac{1}{2}} \doteq \sqrt{r r_1 r_2}\ \exp i\,\frac{\theta+\theta_1+\theta_2}{2}$,

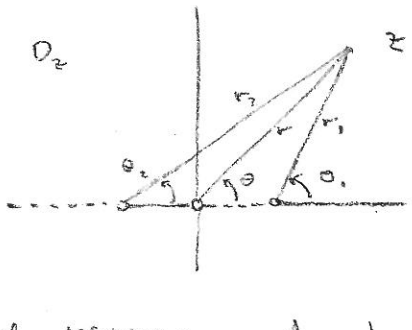
$r, r_1, r_2 > 0$, $-\pi < \theta < \pi$, $-\pi < \theta_1 < \pi$, $-\pi < \theta_2 < \pi$

Forgreningssnit:



Udvidelse af $F(z)$:

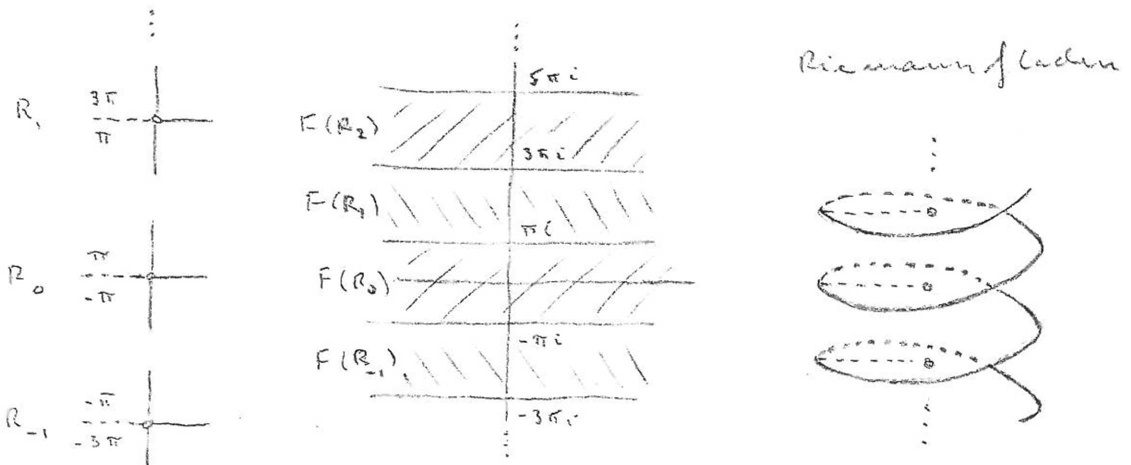
På $\{x \mid -1 < x < 0\}$ sættes $F(z) = -\sqrt{r r_1 r_2}$. Herved opnås, at $F(z)$ bliver analytisk i D_z:



(Ved passage af $\{x \mid -\infty < x < -1\}$ og af $\{x \mid 0 < x < 1\}$ skifter $F(z)$ fortegn.)

99

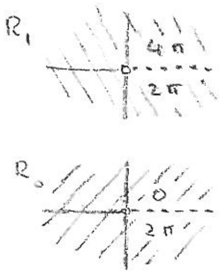
1



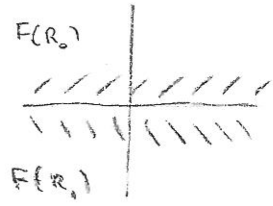
$w = F(z) = \mathrm{Log}\, z$

<u>99</u>

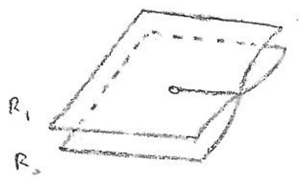
3



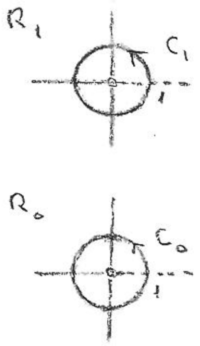
$w = F(z) = z^{\frac{1}{2}}$



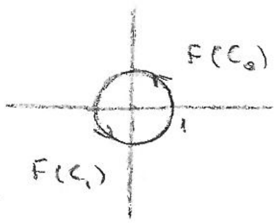
Riemannfladen



4

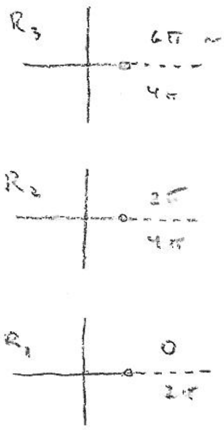


$w = F(z) = z^{\frac{1}{2}}$

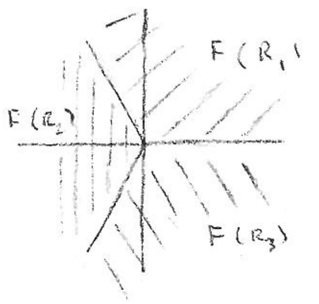


<u>100</u>

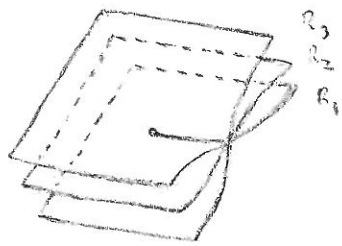
1



$w = (z-1)^{\frac{1}{3}}$
$= re^{i\theta}$

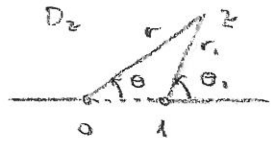


Riemannfladen



3 $\quad w = F(z) = \left(\dfrac{z-1}{z}\right)^{\frac{1}{2}} = \sqrt{\dfrac{r_1}{r}}\ \exp i\ \dfrac{\theta + \theta_1}{2}$, $\quad r, r_1 > 0$, $\ 0 < \theta < 2\pi$, $\ 0 < \theta_1 < 2\pi$

Definitionsområdet for $F(z)$
udvides med $\{x \mid 1 < x < \infty\}$,
jf. opg. <u>99</u> 6 i.

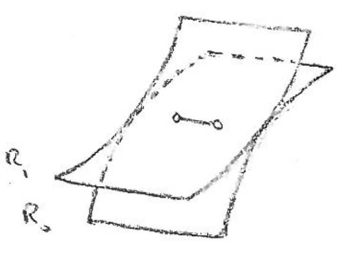


Riemannfladen har to blade

R_1: $\pi + k\,2\pi < \dfrac{\theta + \theta_1}{2} < \pi + (k+1)\,2\pi$

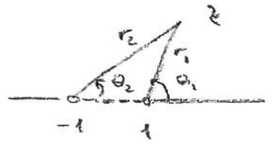
R_0: $k\,2\pi < \dfrac{\theta + \theta_1}{2} < (k+1)\,2\pi$

Riemannfladen



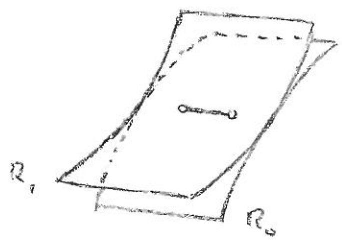
4 $$f(z) = (z^2-1)^{\frac{1}{2}} = \sqrt{r_1 r_2}\ \exp i\ \frac{\theta_1+\theta_2}{2}$$

Definitionmængde



jf. 98 eks. 2

Riemannfladen



$$g(z) = z + f(z)$$

$f_0(z)$: gren af $f(z)$ på R_0

$g_0(z)$ og $g_1(z)$ grene af $g(z)$ på hhv. R_0 og R_1

$g_0(z) = z + (z^2-1)^{\frac{1}{2}}$, hvor o---o ikke overskrides

$g_1(z) = z - (z^2-1)^{\frac{1}{2}}$, do.

$$= \frac{(z-(z^2-1)^{\frac{1}{2}})(z+(z^2-1))^{\frac{1}{2}}}{z+(z^2-1)^{\frac{1}{2}}} = \frac{z^2-(z^2-1)}{z+(z^2-1)^{\frac{1}{2}}} = \frac{1}{g_0(z)}$$

5 (opgave 4 fortsat)

$$g(z) = \sqrt{r_1 r_2}\ \exp i\ \frac{\theta_1}{2}\ \exp i\ \frac{\theta_2}{2}$$

i
$$\left.\begin{aligned} z-1 &= r_1 \exp i\theta_1 \\ z+1 &= r_2 \exp i\theta_2 \end{aligned}\right\} \Rightarrow 2z = r_1 \exp i\theta_1 + r_2 \exp i\theta_2$$

ii
$$g_0(z) = \tfrac{1}{2}(r_1 \exp i\theta_1 + r_2 \exp i\theta_2) + \sqrt{r_1 r_2}\ \exp i\ \frac{\theta_1+\theta_2}{2}$$
$$= \tfrac{1}{2}\left(\sqrt{r_1}\exp i\ \frac{\theta_1}{2} + \sqrt{r_2}\exp i\ \frac{\theta_2}{2}\right)^2$$

iii
$$g_0(z)\overline{g_0(z)} = \tfrac{1}{2}\left(\sqrt{r_1}\exp i\ \frac{\theta_1}{2} + \sqrt{r_2}\exp i\ \frac{\theta_2}{2}\right)^2 \cdot$$
$$\tfrac{1}{2}\left(\sqrt{r_1}\exp\left(-i\frac{\theta_1}{2}\right) + \exp\left(-i\frac{\theta_2}{2}\right)\right)^2$$
$$= \tfrac{1}{4}\left(r_1 + \sqrt{r_1 r_2}\left(\exp i\ \frac{\theta_1-\theta_2}{2} + \exp i\ \frac{\theta_2-\theta_1}{2}\right) + r_2\right)^2$$
$$= \left(\frac{r_1+r_2}{2} + \sqrt{r_1 r_2}\ \cos\frac{\theta_1-\theta_2}{2}\right)^2$$
$$\geq (1+0)^2 = 1$$

$$|g_0(z)| \geq 1$$

fortsættes

5 fortsat

iv $f(z) = \frac{1}{\sqrt{2}} \sqrt{r_1 r_2 + x^2 - y^2 - 1} + i \frac{1}{\sqrt{2}} \sqrt{r_1 r_2 - x^2 + y^2 + 1}$,

hvor $r_1 r_2 = \sqrt{(x^2+y^2+1)^2 - 4x^2}$, jf. opg. 98 2 i

$-1 \leq x \leq 1$, $y = 0$:

$w = g(z) = z + f(z)$

$= x + \frac{1}{\sqrt{2}} \sqrt{r_1 r_2 + x^2 - 1} \pm i \frac{1}{\sqrt{2}} \sqrt{r_1 r_2 - x^2 + 1}$, + for $y \to 0^+$, − for $y \to 0^-$

$r_1 r_2 = \sqrt{(x^2+1)^2 - 4x^2} = 1 - x^2$

$w = x + \frac{1}{\sqrt{2}} \sqrt{1 - x^2 + x^2 - 1} \pm i \frac{1}{\sqrt{2}} \sqrt{1 - x^2 - x^2 + 1}$

$= x \pm i \sqrt{1 - x^2} = u + iv$

$u^2 + v^2 = x^2 + 1 - x^2 = 1$, dvs $|w| = 1$

g afbilder blad R_0 på $|w| > 1$, jf. iii

g afbilder blad R_1 på $|w| < 1$, jf. iii og opg. 4

v $w = z + (z^2 - 1)^{\frac{1}{2}}$

$(w - z)^2 = z^2 - 1$

$w^2 - 2wz + z^2 = z^2 - 1$

$2wz = w^2 + 1$

$z = \frac{1}{2} \left(w + \frac{1}{w} \right)$