16.3 COMPLEX ANALYSIS

Recall that William Rowan Hamilton had by 1837 developed the theory of complex numbers as ordered pairs of real numbers, thus giving one answer to the question of what this mysterious square root of -1 really was. But mathematicians had been using complex numbers since the sixteenth century and even after Hamilton's work did not generally conceive of them in this abstract form. It was the geometrical representation of these numbers, first published by the Norwegian surveyor Caspar Wessel (1745–1818) in an essay in 1797, that ultimately became the basis for a new way of thinking about complex quantities, a way that soon convinced mathematicians that they could use these numbers without undue worry.

16.3.1 Geometrical Representation of Complex Numbers

Wessel's aim in his *On the Analytical Representation of Direction* was not initially related to complex numbers as such. He felt that certain geometrical concepts could be more clearly understood if there was a way to represent both the length and the direction of a line segment in the plane by a single algebraic expression. Wessel made clear that these expressions had to be capable of being manipulated algebraically. In particular, he wanted a way of algebraically expressing an arbitrary change of direction more general than the simple use of a negative sign to indicate the opposite direction.

Wessel began by dealing with addition: "Two straight lines are added if we unite them in such a way that the second line begins where the first one ends and then pass a straight line from the first to the last point of the united lines. This line is the sum of the united lines."[46] Thus, whatever the algebraic expression of a line segment was to be, the addition of two had to satisfy this obvious property drawn from Wessel's conception of motion. In other words, he conceived of line segments as representing vectors. It was multiplication, however, that provided Wessel with the basic answer to his question of the representation of direction. To derive this multiplication, he established a number of properties which he felt were essential. First, the product of two lines in the plane had to remain in the plane. Second, the length of the product line had to be the product of the lengths of the two factor lines. Finally, if all directions were measured from the positive unit line, which he called 1, the angle of direction of the product was to be the sum of the angles of direction of the two factors. Designating by ϵ the line of unit length perpendicular to the line 1, he easily showed that his desired properties implied that $\epsilon^2 = (-\epsilon)^2 = -1$ or that $\epsilon = \sqrt{-1}$. A line of unit length making an angle θ with the positive unit line could now be designated by $\cos\theta + \epsilon \sin\theta$ and, in general, a line of length A and angle θ by $A(\cos\theta + \epsilon\sin\theta) = a + \epsilon b$

where a and b are chosen appropriately (Fig. 16.6). Thus the geometrical interpretation of the complex numbers arose from Wessel's algebraic interpretation of a geometrical line segment. The obvious algebraic rule for addition satisfied Wessel's requirements for that operation, while the multiplication $(a + \epsilon b)(c + \epsilon d) = ac - bd + \epsilon(ad + bc)$ satisfied his axioms for multiplication. Wessel also easily derived from his definitions the standard rules for division and root extraction of complex numbers.

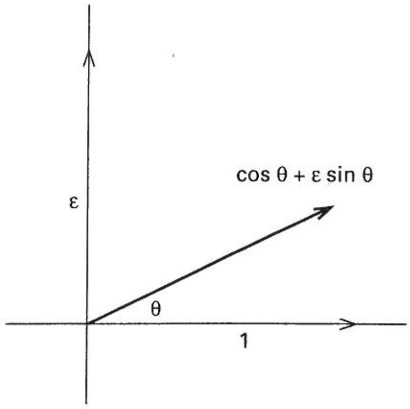

FIGURE 16.6
Wessel's geometric interpretation of complex numbers.

Unfortunately, Wessel's essay remained unread in most of Europe for many years after its publication. The same fate awaited the similar geometric interpretation of the complex numbers put forth by the Swiss bookkeeper Jean-Robert Argand (1768–1822) in a small book published in 1806. This interpretation gained acceptance in the mathematical community only because Gauss used the same geometric interpretation of the complex numbers in his proofs of the fundamental theorem of algebra and in his study of quartic residues (Fig. 16.7). Gauss was so intrigued with the fundamental theorem—that every polynomial $p(x)$ with real coefficients has a real or complex root—that he published four different proofs of it, in 1799, 1815, 1816, and 1848. Each proof used in some form or other the geometric interpretation of complex numbers, although in the first three proofs Gauss hid this notion by considering the real and imaginary parts of the numbers separately. Thus, in his initial proof, Gauss in essence set $p(x + iy) = u(x, y) + iv(x, y)$ and then noted that a root of p would be an intersection point of the curves $u = 0$ and $v = 0$. He therefore made a detailed study of these curves and, through the use of the intermediate value theorem, showed that the curves must cross. It was only in his final proof in 1848 that Gauss believed mathematicians would be comfortable enough with the geometric interpretation of complex numbers so that he could use it explicitly. In fact, in that proof, similar to his first one, he even permitted the coefficients of the polynomial to be complex.

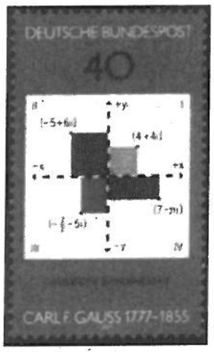

FIGURE 16.7
The Gaussian complex plane on a German stamp.

16.3.2 Complex Integration

By the second decade of the century, Gauss, with his clear understanding of the meaning of complex numbers, began to develop of the theory of complex functions. In a letter of 1811 to his friend Friedrich Wilhelm Bessel (1784–1846), Gauss not only discussed the geometric interpretation of the complex numbers but also discussed the meaning of $\int_{\mu}^{\nu} \phi(x)\,dx$ where the variable x *is* complex:

> We must assume that x passes through infinitely small increments (each of the form $\alpha + \beta i$) from the value for which the integral is 0 to $x = a + bi$, and then sum all the $\phi(x)\,dx$. In this way the meaning is completely established. But the passage can occur in infinitely many ways; just as one can think of the entire domain of all real magnitudes as an infinite straight line, so one can make the entire domain of all magnitudes, real and imaginary, meaningful as an infinite plane, wherein each point determined by abscissa $= a$ and ordinate $= b$ represents the magnitude $a + bi$ as it were. The continuous passage from one value of x to another $a + bi$ accordingly occurs along a line and is consequently possible in infinitely many ways.[47]

Gauss went on to assert the "very beautiful theorem" that as long as $\phi(x)$ is never infinite within the region enclosed by two different curves connecting the starting and ending points of this integral, then the value of the integral is the same along both curves. Although he did not express himself in those terms, Gauss was considering $\phi(x)$ as an analytic function. In any case, he never published a proof of this result. Such a proof was published in 1825 by Cauchy, however, so the theorem is generally called Cauchy's integral theorem.

Cauchy first considered the question of integration in the complex domain in a memoir written in 1814 but not published until 1827. In this work he was mainly interested in the evaluation of definite integrals where one or both of the limits of integration is infinite. To perform such an evaluation, he attempted to make rigorous various procedures developed by Euler and Laplace involving moving the paths of integration into the complex plane. In particular, he used an idea of Euler's to derive the Cauchy-Riemann equations. Euler, in a paper written about 1777, asserted that the most important theorem about complex functions was that every function $Z(x + iy)$ that can be written as the sum $M(x, y) + iN(x, y)$ has the property that $Z(x - iy) = M - iN$. In this case it follows that if

$$V = \int Z\,dz = \int (M + iN)(dx + i\,dy) = \int M\,dx - N\,dy + i \int N\,dx + M\,dy = P + iQ,$$

then, replacing $x + iy$ by $x - iy$,

$$P - iQ = \int (M - iN)(dx - i\,dy)) = \int M\,dx - N\,dy - i \int N\,dx + M\,dy.$$

Therefore $P = \int M\,dx - N\,dy$ and $Q = \int N\,dx + M\,dy$, where, as usual for Euler, the integral signs stand for antidifferentiation. Because P is the integral of the differential $M\,dx - N\,dy$ it follows that

$$\frac{\partial M}{\partial y} = -\frac{\partial N}{\partial x}.$$

Similarly, the expression for Q shows that

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}.$$

These two equations, the Cauchy-Riemann equations, ultimately became the characteristic property of complex functions.

In his 1821 *Cours d'analyse*, Cauchy dealt with complex quantities, as had Euler, by considering separately the real and imaginary parts. Thus he considered the "symbolic expressions" $a + ib$ and multiplied them together using normal algebraic rules "as if $\sqrt{-1}$ was a real quantity whose square was equal to -1."[48] He defined a function of a complex variable in terms of two real functions of two real variables and showed what is meant by the various standard transcendental functions in the complex domain. He then generalized most of his results on convergence of series to complex numbers by using the modulus $\sqrt{a^2 + b^2}$ of the quantity $z = a + ib$ as the analogue of the absolute value of a real number. He also defined continuity for a complex function in terms of the continuity of its two constituent functions.

Not until 1825, however, having discovered his new definition of a definite integral, was Cauchy able to deal with complex functions in their own right. In his *Mémoire sur les intégrales définies prises entre des limites imaginaires* (*Memoir on definite integrals taken between imaginary limits*), he explicitly defined the definite complex integral

$$\int_{a+ib}^{c+id} f(z)\, dz$$

to be the "limit or one of the limits to which the sum of products of the form $[(x_1 - a) + i(y_1 - b)]f(a + ib)$, $[(x_2 - x_1) + i(y_2 - y_1)]f(x_1 + iy_1), \ldots, [(c - x_{n-1}) + i(d - y_{n-1})]f(x_{n-1} + iy_{n-1})$ converge when each of the two sequences $a, x_1, x_2, \ldots, x_{n-1}, c$ and $b, y_1, y_2, \ldots, y_{n-1}, d$ consist of terms that increase or decrease from the first to the last and approach one another indefinitely as their number increases without limit."[49] In other words, Cauchy directly generalized his definition of a real definite integral by simply taking partitions of the two intervals $[a, b]$ and $[c, d]$. Cauchy realized, however, as had Gauss, that there were infinitely many different paths of integration beginning at $a + ib$ and ending at $c + id$. It was therefore not clear that this definition made sense. To demonstrate his integral theorem, which in effect stated that the definition did make sense, he began by considering a path determined by the parametric equations $x = \phi(t)$, $y = \psi(t)$, where ϕ and ψ are monotonic differentiable functions of t in the interval $[\alpha, \beta]$, with $\phi(\alpha) = a$, $\phi(\beta) = c$, $\psi(\alpha) = b$, and $\psi(\beta) = d$. The two sequences $\{x_j\}$ and $\{y_j\}$ are then determined by taking a single sequence $\alpha, t_1, t_2, \ldots, t_{n-1}, \beta$ and calculating the values of this sequence under ϕ and ψ, respectively. Assuming that the lengths of the various subintervals determined by the t_j are small, Cauchy noted that $x_j - x_{j-1} \approx (t_j - t_{j-1})\phi'(t_j)$ and $y_j - y_{j-1} \approx (t_j - t_{j-1})\psi'(t)$. It follows that the definite integral is the limit of sums of terms of the form $(t_j - t_{j-1})[\phi'(t_j) + i\psi'(t_j)]f[\phi(t_j) + i\psi(t_j)]$ and therefore can be rewritten in the form

$$\int_{a+ib}^{c+id} f(z)\, dz = \int_{\alpha}^{\beta} [\phi'(t) + i\psi'(t)]f[\phi(t) + i\psi(t)]\, dt,$$

or, setting $x' = \phi'(t)$, $y' = \psi'(t)$, as

$$\int_{\alpha}^{\beta} (x' + iy')f(x + iy)\, dt.$$

"Now suppose that the function $f(x + iy)$ remains bounded and continuous as long as x stays between the limits a and c, and y between the limits b and d. In this special case one easily proves that the value of the integral... is independent of the nature of the functions $x = \phi(t)$, $y = \psi(t)$."[50] Cauchy's proof of this statement, which requires the existence and continuity of $f'(z)$—and Cauchy had not explicitly defined what was meant by the derivative of a complex function—was based on the calculus of variations. Cauchy varied the curve infinitesimally by replacing the functions ϕ and ψ by $\phi + \epsilon u$, $\psi + \epsilon v$, where ϵ is "an infinitesimal of the first order," and u, v both vanish at $t = \alpha$ and $t = \beta$, and expanded the corresponding change in the integral in a power series in ϵ. Using an integration by parts, Cauchy demonstrated that the coefficient of ϵ in this series is 0 and therefore that an infinitesimal change in the path of integration produces an infinitesimal change in the integral of the order of ϵ^2. Cauchy concluded that a finite change in the path, that is, a change from one path of integration to a second such path, can produce but an infinitesimal change in the integral, that is, no change at all. The integral theorem was therefore proved according to Cauchy's, if not modern, standards.

Cauchy next considered the case where f becomes infinite at some value $z_1 = r + is$ in the rectangle $a \le x \le c, b \le y \le d$. The integrals along two paths that together enclose z_1 are no longer the same. Defining R to be $\lim_{z \to z_1}(z - z_1)f(z)$, Cauchy calculated the difference in the integrals along two paths infinitely close to each other and to the point z_1 to be $2\pi Ri$. For example, if $f(z) = 1/(1 + z^2)$, then f becomes infinite at $z = i$. Because

$$\lim_{z \to i} \frac{z - i}{1 + z^2} = \lim_{z \to i} \frac{z - i}{(z - i)(z + i)} = \frac{1}{2i},$$

it follows that the difference in the values of the integrals of this function over the two paths L_1 and L_2 from -2 to 2 in Fig. 16.8 is

$$2\pi \frac{1}{2i} i = \pi.$$

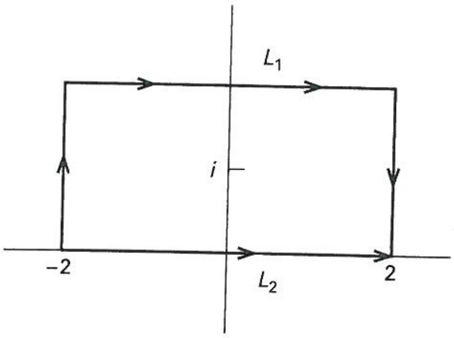

FIGURE 16.8
Two integration paths for $f(z) = \frac{1}{1+z^2}$ from -2 to 2.

In a paper written in 1826, Cauchy generalized his integral theorem somewhat. Given a value z_1 for which $f(z)$ is infinite, Cauchy noted that the expansion of $f(z_1 + \epsilon)$ in powers of ϵ will begin with negative powers. The coefficient of $1/\epsilon$ in this expansion is what Cauchy terms the **residue** of $f(z)$ at z_1, denoted by $R(f, z_1)$. Thus, if $(z - z_1)f(z) = g(z)$ is bounded near z_1, then

$$f(z_1 + \epsilon) = \frac{g(z_1 + \epsilon)}{\epsilon} = \frac{1}{\epsilon} g(z_1) + g'(z_1 + \theta\epsilon)$$

for θ a number between 0 and 1. It follows that the residue of $f(z)$ at z_1 is $g(z_1)$, the same value denoted earlier by R.

Cauchy noted that his theory of residues had applications to such problems as the splitting of rational fractions, the determination of the values of certain definite integrals, and the solution of certain types of equations. For example, he demonstrated that

$$\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2}\, dx = \pi e^{-1}$$

by extending the interval of integration to a closed path in the complex plane containing the value i for which the integrand becomes infinite. The central idea in this calculation is that the integral over the path consisting of a half circle and an interval on the real line can be calculated by means of residues, but as the radius of the half circle (and the length of the interval) get larger, the part of the integral taken over the half circle approaches 0.

16.3.3 Complex Functions and Line Integrals

There are many other standard results in complex function theory for which Cauchy was at least partially responsible, most being applications of his integral theorem or his calculus of residues. But the discussion of his work will be concluded with a brief analysis of a paper of 1846 which, although it did not mention complex functions at all, led to a new way of proving the integral theorem and also provided the beginning of some fundamental ideas in both vector analysis and topology. This short paper, *Sur les intégrales qui s'étendent a tous les points d'une courbe fermée* (*On the integrals which extend to all the points of a closed curve*) contained the bare statement of several theorems, without proofs. Cauchy promised to provide the proofs later, but apparently did not do so. The theorems deal with a function k of several variables $x, y, z, \ldots$ that is to be integrated along the boundary curve Γ of a surface S lying in a space of an unspecified number of dimensions. The most important results are collected in the following

THEOREM *Suppose*

$$k = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds} + \cdots$$

where $X\,dx + Y\,dy + Z\,dz + \cdots$ *is an exact differential. (To say that this differential is exact is to say that* $\partial X/\partial y = \partial Y/\partial x$, $\partial X/\partial z = \partial Z/\partial x$, $\partial Y/\partial z = \partial Z/\partial y, \ldots$.) *Suppose that the function* k *is finite and continuous everywhere on* S *except at finitely many points* $P, P', P'', \ldots$ *in its interior. If* $\alpha, \beta, \gamma, \ldots$ *are closed curves in* S *surrounding these points respectively, then*

$$\int_\Gamma k\,ds = \int_\alpha k\,ds + \int_\beta k\,ds + \int_\gamma k\,ds + \cdots.$$

In particular, if there are no such singular points, then

$$\int_\Gamma k\,ds = 0.$$

In the two-dimensional case, where S is a region of the plane and k is an arbitrary differential, then

$$\int_\Gamma k\,ds = \pm \int\int_S \left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) dx\,dy.$$

If k is an exact differential, then $\partial X/\partial y = \partial Y/\partial x$, so the right side, and therefore the left, vanish.

The Cauchy integral theorem follows from the last statement. A complex function $f(z) = f(x + iy)$ can be expressed as $f(x, y) = u(x, y) + iv(x, y)$ and, therefore, since $dz = dx + i\,dy$,

$$\int f(z)\,dz = \int (u\,dx - v\,dy) + i \int (v\,dx + u\,dy).$$

The Cauchy-Riemann equations then imply that both integrands are exact differentials and therefore that the integral theorem holds.

More interesting than the integral theorem, however, is the appearance in Cauchy's paper both of the concept of a line integral in n-dimensional space (and of the matter-of-fact occurrence of a space of dimension higher than three) and of the statement (in the next to the last sentence) of the theorem today generally known as Green's theorem. In fact, results somewhat akin to that theorem appear in an 1828 paper of George Green (1793–1841) dealing with electricity and magnetism, but Cauchy's version is the first printed statement of the result so named in today's textbooks. Finally, the expression of the line integral around the boundary of the surface as a sum of line integrals around isolated singular points, whose values are called **periods**, marked the beginning of the study of the relationships of integrals to surfaces over which they are not defined everywhere. Since Cauchy never published the proof of his 1846 theorem, one can only speculate as to how far he carried all of these new concepts. It was Riemann, however, who restated Cauchy's results a few years later, with full proofs, and extended the result on periods far beyond Cauchy's conception.

16.3.4 Riemann and Complex Functions

Riemann's dissertation, *Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse* (*Foundations for a general theory of functions of one complex variable*), began with a discussion of an important distinction between real and complex functions. Although the definition of function, "to every one of [the] values [of a variable quantity z] there corresponds a single value of the indeterminate quantity w,"[51] can be applied both to the real and the complex case, Riemann realized that in the latter case, where $z = x + iy$ and $w = u + iv$, the limit of the ratio dw/dz defining the derivative could well depend on how dz approaches 0. Because for functions defined algebraically one could calculate the derivative formally and not have this problem, Riemann decided to make this existence of the derivative the basis for the concept of a complex function: "The complex variable w is called a function of another complex variable z when its variation is such that the value of the derivative dw/dz is independent of the value of dz."[52] Cauchy, of

Biography

GEORG BERNHARD RIEMANN (1826–1866)

Riemann needed his father's permission to switch from the study of theology and philology to the study of mathematics in 1846 when he enrolled at the University of Göttingen. He had started life in the village of Breselenz, about 60 miles southeast of Hamburg, and now he would journey to Berlin because mathematics education was not particularly strong at Göttingen. In Berlin he met Dirichlet, who became his mentor. He returned to Göttingen a few years later to study with Gauss and received his Ph.D. in 1851. For two years he researched and prepared his lectures for his *Habilitation* to qualify to teach at Göttingen. In 1857 he was appointed as an associate professor and two years later, on the death of Dirichlet, who had in the meantime come to Göttingen, as full professor. His mathematical work was brilliant, but tuberculosis cut his work short when it claimed his life in the summer of 1866 during one of his several trips to Italy to find a cure.

course, had essentially used this notion in his entire discussion of complex functions but had only made it explicit toward the end of his career.

As a first application of this definition, Riemann showed that such a complex function considered as a mapping from the z-plane to the w-plane preserves angles. For suppose p' and p'' are infinitely close to the origin P in the z-plane, with their images q', q'' infinitely close to the image Q of P. Writing the infinitesimal distance from p' to P both as $dx' + i\,dy'$ and as $\epsilon' e^{i\phi'}$, and that from q' to Q as both $du' + i\,dv'$ and $\eta' e^{i\psi'}$, with similar notations for the other infinitesimal distances, Riemann noted that his condition on the function implies that

$$\frac{du' + i\,dv'}{dx' + i\,dy'} = \frac{du'' + i\,dv''}{dx'' + i\,dy''}$$

or that

$$\frac{du' + i\,dv'}{du'' + i\,dv''} = \frac{\eta'}{\eta''} e^{i(\psi' - \psi'')} = \frac{dx' + i\,dy'}{dx'' + i\,dy''} = \frac{\epsilon'}{\epsilon''} e^{i(\phi' - \phi'')}.$$

It follows that $\eta'/\eta'' = \epsilon'/\epsilon''$ and that $\psi' - \psi'' = \phi' - \phi''$, or, in other words, that the infinitesimal triangles $p'Pp''$ and $q'Qq''$ are similar. Such an angle-preserving mapping is called a **conformal** mapping. In some sense, both Euler and Gauss knew that analytic complex functions had this property, but it was Riemann who gave this argument and who, in addition, was able to demonstrate the Riemann mapping theorem, that any two simply connected regions in the complex plane can be mapped conformally on each other by means of a suitably chosen complex function.

Riemann next derived the Cauchy-Riemann equations by determining what the existence of the derivative means in terms of the two functions u and v:

$$\frac{dw}{dz} = \frac{du + i\,dv}{dx + i\,dy} = \frac{\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + i\left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy\right)}{dx + i\,dy}$$

$$= \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)dx + \left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right)i\,dy}{dx + i\,dy}.$$

If this value is independent of how dz approaches 0, then setting dx and dy in turn equal to zero and equating the real and imaginary parts of the two resulting expressions shows that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Conversely, if those Cauchy-Riemann equations are satisfied, then the desired derivative is easily calculated to be $\partial u/\partial x + i\,\partial v/\partial x$, a value independent of dz. Riemann made these equations the center of his theory of complex functions, along with the second set of partial differential equations easily derived from them:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

As an example, Riemann gave a detailed proof of the Cauchy integral theorem following the outline provided by Cauchy in 1846. The important idea was Green's theorem, which Riemann stated in the following form:

THEOREM *Let X and Y be two functions of x and y continuous in a finite region T with infinitesimal area element designated by dT. Then*

$$\int_T \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) dT = -\int_S (X\cos\xi + Y\cos\eta)\,ds$$

where the latter integral is taken over the boundary curve S of T, ξ, η designating the angles the inward pointing normal line to the curve makes with the x- and y-axis respectively.

Riemann proved this by using the fundamental theorem of calculus to integrate $\partial X/\partial x$ along lines parallel to the x-axis, getting values of X where the lines cross the boundary of the region. Because $dy = \cos\xi\,ds$ at each of those points, he could integrate with respect to y to get

$$\int\left[\int \frac{\partial X}{\partial x}dx\right]dy = -\int X\,dy = -\int X\cos\xi\,ds.$$

The other half of the theorem is proved similarly. Riemann then noted that

$$\frac{dx}{ds} = \pm\cos\eta \quad \text{and} \quad \frac{dy}{ds} = \mp\cos\xi$$

where the sign depends on whether one gets from the tangent line to the inward normal line by traveling counterclockwise or clockwise. It follows that Green's theorem can be rewritten as

$$\int_T \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} \right) dT = \int_S \left(X\frac{dy}{ds} - Y\frac{dx}{ds} \right) ds,$$

from which the Cauchy integral theorem follows easily.

Much of Riemann's dissertation involved the introduction of an entirely new concept in the study of complex functions, the idea of a Riemann surface. In the case of functions of a real variable, it is possible to picture the function by a curve in two-dimensional space. Such a representation is no longer possible for complex functions, because the graph would need to be in a space of four real dimensions. An alternative way of picturing complex functions, then, is to trace the independent variable z along a curve in one plane and consider the curve generated by the dependent variable w in another plane. Riemann realized from the fact that a complex function always had a power series representation that "a function of $x + iy$ defined in a region of the (x, y) plane can be continued analytically in only one way." It follows that once one knows the values in a certain region, one can continue the function and even return to the same z value by, say, a continuous curve. There are then two possibilities. "Depending on the nature of the function to be continued, either this function will always assume the same value for the same value of [z], no matter how it is continued, or it will not."[53] In the first case, Riemann called the function single-valued, while in the second it is multiple-valued. As a simple example of the latter, one can take $w = z^{1/2}$. To study such functions effectively, it was not possible simply to use two planes as indicated above, for one would not know which value the function had for a given point on the first plane. Thus Riemann came up with a new idea, to use a multiple plane, a covering of the z-plane by as many sheets as the function has values. These sheets are attached along a line, say the negative real axis, in such a way that whenever one moves in a curve across that line one changes from one sheet to another. In this way the multiple-valued function has only one value defined at each point of this Riemann surface. Since it may happen that after several circuits (two in the example above) one returns to a former value, the top sheet of this covering must be attached to the bottom one. It follows that it is not in general possible to construct a physical model of a Riemann surface in three-dimensional space. Nevertheless, the study of Riemann surfaces, initiated by Riemann to deal with multiple-valued complex functions, soon led Riemann and others into the realm of what is today called **topology**. The connection of topology with integration along curves and surfaces, barely touched by Cauchy in 1846, was explored in great detail in the second half of the nineteenth century and the early years of the twentieth.