6.2

3 $X \sim b(n, p)$, $p = P(A)$

Estimator for p : $\hat{p} = \frac{X}{n}$

i $E\hat{p} = \frac{EX}{n} = \frac{np}{n} = p$, dvs. $\hat{p}$ er central estimator

ii $\mathrm{Var}\,\hat{p} = \frac{\mathrm{Var}\,X}{n^2} = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n} \to 0$ for $n \to \infty$,

dvs. $\hat{p}$ er konsistent estimator, jf. sætn. 6.1 side 297

iii $\sigma_{\hat{p}} = \sqrt{\mathrm{Var}\,\hat{p}} = \sqrt{\frac{p(1-p)}{n}}$

5 N : Størrelse af fiskepopulation, hvoraf k er mærkede.

a $X \sim h(k, N, n)$ (notation som i noten 'Summationer' med $M := k$)

$EX = n\frac{k}{N}$

Sæt $n\frac{k}{N} = x \iff N = \frac{nk}{x}$

Forslag til estimator : $\hat{N} = \frac{nk}{X}$

$E\hat{N} = nk\,E\left[\frac{1}{X}\right] \neq N$, dvs. $\hat{N}$ er ikke-central estimator

b $X \sim g\left(\frac{k}{N}\right)$ (notation som i noten 'Summationer' med $p := \frac{k}{N}$)

$EX = \frac{1}{\frac{k}{N}} = \frac{N}{k}$

Sæt $\frac{N}{k} = x \iff N = kx$

Forslag til estimator : $\hat{N} = kX$

$E\hat{N} = k\,EX = k\frac{N}{k} = N$, dvs. $\hat{N}$ er central estimator

c Fuldstændig opblanding af de mærkede fisk

7 X: # begivenheder i interval af længde t

$X \sim p(\lambda t)$, $\hat{\lambda} = \frac{X}{t}$

i $E\hat{\lambda} = \frac{EX}{t} = \frac{\lambda t}{t} = \lambda$, dvs $\hat{\lambda}$ er central estimator

ii $\mathrm{Var}\,\hat{\lambda} = \frac{\mathrm{Var}\,X}{t^2} = \frac{\lambda t}{t^2} = \frac{\lambda}{t} \Rightarrow \sigma_{\hat{\lambda}} = \sqrt{\frac{\lambda}{t}}$

6.2

14 $X \sim N(\mu, \sigma^2)$

Observationer : 7,6 9,6 10,4 10,7 11,9 14,1 14,6 18,5

Beregning : $\bar{x} = 12,175$, $s = 3,4346$

a $p = P(X \geq 20) = 1 - \Phi\left(\frac{20-\mu}{\sigma}\right)$

Forslag til estimat af p : $1 - \Phi\left(\frac{20-\bar{x}}{s}\right)$

Estimat : $\hat{p} = 1 - \Phi\left(\frac{20-12,175}{3,4346}\right) = 1 - \Phi(2,2786)$

$= 1 - 0,9887 = 0,0113$

b $P(X \geq x) = 0,95 \iff \frac{x-\mu}{\sigma} = \Phi^{-1}(0,95) = 1,645$

$\iff x = 1,645\,\sigma + \mu$

Estimat : $\hat{x} = 1,645\,s + \bar{x} = 1,645 \cdot 3,4346 + 12,175$

$= 17,82$

6.3

17 Observationer af IQ , $X_A \sim N(\mu_A, \sigma^2)$, $X_B \sim N(\mu_B, \sigma^2)$

A : 106 114 116 123 124 133 $n_A = 6$ $\bar{x}_A = 119,33$

B : 99 113 114 121 126 $n_B = 5$ $\bar{x}_B = 114,60$

Spredning i begge observationsrækker : $\sigma = 15$

Konfidensinterval for $\mu_A - \mu_B$ med konfidensgrad 0,95 :

$\mu_A - \mu_B = \bar{x}_A - \bar{x}_B \pm z_{0,975}\,\sigma\sqrt{\frac{1}{n_A} + \frac{1}{n_B}}$

$= 119,33 - 114,60 \pm 1,960 \cdot 15\sqrt{\frac{1}{6} + \frac{1}{5}}$

$= 4,73 \pm 17,80$

21 Konfidensinterval for p med konfidensgrad 0,95 ,
intervallængde $\leq 0,1$ $(\pm 0,05)$:

$z_{0,975}\sqrt{\frac{p(1-p)}{n}} \leq 0,05$

fortsættes

6.2

21 fortsat

a $p \le 0{,}2$

$$1{,}960 \sqrt{\frac{0{,}2 \cdot 0{,}8}{n}} \le 0{,}05 \iff n \ge 245{,}9 \Rightarrow n \ge 246$$

b $p(1-p)$ har maksimum $\frac{1}{4}$ for $p = \frac{1}{2}$:

$$1{,}960 \sqrt{\frac{\frac{1}{4}}{n}} \le 0{,}05 \iff n \ge 384{,}1 \Rightarrow n \ge 385$$

c $\pm \frac{1}{2}\, 1{,}960 \sqrt{\frac{1}{n}} \approx \pm \frac{1}{\sqrt{n}}$

26 $S \sim N\left(\sigma, \frac{\sigma^2}{2n}\right)$ app.

Konfidensinterval for middelværdien σ med konfidensgrad q :

$$\sigma = s \pm z_{\frac{1+q}{2}} \frac{\sigma}{\sqrt{2n}} \iff \sigma \left(1 \pm \frac{z_{\frac{1+q}{2}}}{\sqrt{2n}}\right) = s$$

$$\iff \sigma = \frac{s}{1 \pm \frac{z_{\frac{1+q}{2}}}{\sqrt{2n}}}$$

27 $X \sim p(\lambda t)$

$\hat{\lambda} = \frac{X}{t}$, $E\hat{\lambda} = \lambda$, $\sigma_{\hat{\lambda}} = \sqrt{\frac{\lambda}{t}}$, jf. opg. 6.2 7

$X \sim N\left(\lambda, \frac{\lambda}{t}\right)$ app.

Yderligere approksimation : $\frac{\lambda}{t} \approx \frac{\hat{\lambda}}{t} = \frac{\frac{x}{t}}{t} = \frac{x}{t^2}$

$X \sim N\left(\lambda, \frac{x}{t^2}\right)$ app.

Konfidensinterval for middelværdien λ med konfidensgrad q :

$$\lambda = \hat{\lambda} \pm z_{\frac{1+q}{2}} \sqrt{\frac{x}{t^2}} = \frac{x}{t} \pm z_{\frac{1+q}{2}} \frac{\sqrt{x}}{t}$$

6.4

33 $X \sim b(n, p)$

$$L(p) = \binom{n}{x} p^x (1-p)^{n-x}$$

fortsættes

33 fortsat

$$l(p) = \ln\binom{n}{x} + x\ln p + (n-x)\ln(1-p)$$

$$\frac{dl}{dp} = \frac{x}{p} + \frac{n-x}{1-p}(-1) = 0 \iff x(1-p) = (n-x)p$$

$$\iff x = np \iff p = \frac{x}{n}$$

$$\left.\frac{d^2l}{dp^2}\right|_{p=\frac{x}{n}} = -\frac{x}{p^2} + \left.\frac{n-x}{(1-p)^2}(-1)\right|_{p=\frac{x}{n}} = -\frac{n^2}{x} - \frac{n^2}{n-x} < 0$$

MLE: $\hat{p} = \frac{X}{n}$

34 $X_1, X_2, \dots, X_n \sim U[a;b]$ uafhængige, b ukendt

$f(x) = \frac{1}{b-a}$, $a < x < b$

i $L(b) = \frac{1}{(b-a)^n}$

$l(b) = -n\ln(b-a)$

$\frac{dl}{db} = -\frac{n}{b-a} < 0 \Rightarrow l(b)$ er aft. fkt. af b

$\Rightarrow b$ skal vælges som $x_{(n)}$

MLE $\hat{b} = X_{(n)}$

ii $EX = \frac{a+b}{2}$

Sæt $\frac{a+b}{2} = \bar{x} \Rightarrow b = 2\bar{x} - a$

Estimator: $\tilde{b} = 2\bar{X} - a$

37 $X_1, X_2, \dots, X_n$ uafhængige og identisk fordelte

$f(x) = e^{-(x-\theta)}$, $x > \theta$

i $L(\theta) = e^{-(\Sigma x_i - n\theta)}$

$l(\theta) = -\Sigma x_i + n\theta$

$\frac{dl}{d\theta} = n > 0 \Rightarrow l(\theta)$ voks. fkt af θ

$\Rightarrow \theta$ skal vælges som $x_{(1)}$

MLE: $\hat{\theta} = X_{(1)}$

fortsættes

6.4

37 fortsat

ii $X_1 = X - \theta$, $EX_1 = \frac{1}{1} = 1$, $EX = EX_1 + \theta = 1 + \theta$

Sæt $1 + \theta = \bar{x} \Rightarrow \theta = \bar{x} - 1$

Estimator : $\tilde{\theta} = \bar{X} - 1$

38 $X_1, X_2, \ldots, X_n$ uafhængige og identisk fordelte

$f(x) = \theta x^{\theta - 1}$, $0 < x < 1$

a $EX = \int_0^1 x \, \theta x^{\theta-1} dx = \theta \int_0^1 x^\theta dx = \theta \left[\frac{x^{\theta+1}}{\theta+1} \right]_0^1 = \frac{\theta}{\theta+1}$

Sæt $\frac{\theta}{\theta+1} = \bar{x} \Rightarrow (1-\bar{x})\theta = \bar{x} \Rightarrow \theta = \frac{\bar{x}}{1-\bar{x}}$

Estimator : $\tilde{\theta} = \frac{\bar{X}}{1-\bar{X}}$

b $L(\theta) = \theta^n \left(\prod x_i \right)^{\theta-1}$

$\ell(\theta) = n \ln \theta + (\theta - 1) \sum \ln x_i$

$\frac{d\ell}{d\theta} = \frac{n}{\theta} + \sum \ln x_i = 0 \Rightarrow \theta = -\frac{n}{\sum \ln x_i}$

$\left.\frac{d^2\ell}{d\theta^2}\right|_{\theta = -\frac{n}{\sum \ln x_i}} = \left. -\frac{n}{\theta^2} \right|_{\theta = -\frac{n}{\sum \ln x_i}} = -\frac{(\sum \ln x_i)^2}{n} < 0$

MLE : $\hat{\theta} = -\frac{n}{\sum \ln X_i}$

c $\hat{\theta} \sim N\left(\theta, \frac{1}{n I(\theta)}\right)$ approx., jf. side 321 nederst

$n I(\theta) = -E\left[\frac{d^2\ell}{d\theta^2}\right] = -E\left[-\frac{n}{\theta^2}\right] = \frac{n}{\theta^2}$

Yderligere approksimation : $I(\theta) \approx I(\hat{\theta}) = \frac{1}{\hat{\theta}^2}$

Konfidensinterval for middelværdien θ med konfidensgrad q :

$\theta = \hat{\theta} \pm z_{\frac{1+q}{2}} \frac{\hat{\theta}}{\sqrt{n}} = \hat{\theta}\left(1 \pm z_{\frac{1+q}{2}} \frac{1}{\sqrt{n}}\right)$