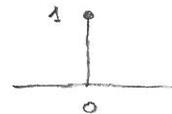


132

$$a \quad G(s) = 1 \quad r_0 = 1$$

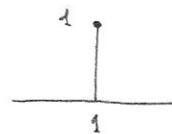
$$G^{(j)}(s) = 0 \quad r_j = 0, \quad j = 1, 2, \dots$$



$$b \quad G(s) = s \quad r_0 = 0$$

$$G'(s) = 1 \quad r_1 = 1$$

$$G^{(j)}(s) = 0 \quad r_j = 0, \quad j = 2, 3, \dots$$

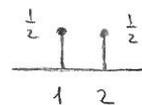


$$c \quad G(s) = \frac{1}{2}(s + s^2) \quad r_0 = 0$$

$$G'(s) = \frac{1}{2} + s \quad r_1 = \frac{1}{2}$$

$$G''(s) = 1 \quad r_2 = \frac{1}{2}$$

$$G^{(j)}(s) = 0 \quad r_j = 0, \quad j = 3, 4, \dots$$



133

$$a \quad X \sim g(r)$$

$$G_X(s) = \sum_{k=1}^{\infty} s^k r(1-r)^{k-1} = rs \sum_{k=1}^{\infty} (s(1-r))^{k-1}$$

$$= rs \sum_{k=0}^{\infty} (s - rs)^k = rs \frac{1}{1 - (s - rs)} = \frac{rs}{1 - s + rs}$$

$$b \quad Y \sim nb(r, r), \quad Y = \sum_{j=1}^r X_j, \quad X_j \sim g(r), \quad j = 1, \dots, r$$

$n = h.$

$$G_Y(s) = (G_X(s))^r = \left(\frac{rs}{1 - s + rs} \right)^r$$

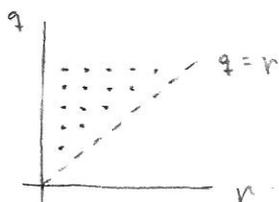
134

X, Y pos. heltallige, n, h , $r, q \in \mathbb{N}$, $q > r$

$$P\left(\frac{X}{X+Y} = \frac{r}{q}\right) = \sum_{q=1}^{\infty} \sum_{r=1}^{q-1} P(X=r, X+Y=q)$$

$$P(X=r, X+Y=q) = P(X=r, Y=q-r) = P(X=r) P(Y=q-r)$$

$$\sum_{q=1}^{\infty} \sum_{r=1}^{q-1} s^{q-1} r P(X=r, X+Y=q) = \sum_{q=1}^{\infty} \sum_{r=1}^{q-1} r s^{q-1} P(X=r) s^{q-r} P(Y=q-r)$$



$$= \sum_{r=1}^{\infty} r s^{r-1} P(X=r) \sum_{q=r+1}^{\infty} s^{q-r} P(Y=q-r)$$

$$= G'_X(s) \sum_{q=1}^{\infty} s^q P(Y=q)$$

$$= G'_X(s) G_Y(s)$$

fortsatt

$$\begin{aligned}
 \int_0^1 G_X'(s) G_Y'(s) ds &= \sum_{q=1}^{\infty} \sum_{r=1}^{q-1} \left(\int_0^1 s^{q-1} ds \right) \cdot P\left(\frac{X}{X+Y} = \frac{r}{q}\right) \\
 &= \sum_{q=1}^{\infty} \sum_{r=1}^{q-1} \left[\frac{s^q}{q} \right]_0^1 \cdot P\left(\frac{X}{X+Y} = \frac{r}{q}\right) \\
 &= \sum_{q=1}^{\infty} \sum_{r=1}^{q-1} \frac{r}{q} P\left(\frac{X}{X+Y} = \frac{r}{q}\right) = E\left[\frac{X}{X+Y}\right]
 \end{aligned}$$

also $E\left[\frac{X}{X+Y}\right] = \int_0^1 G_X'(s) G_Y'(s) ds$

137

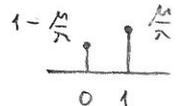
X_1, X_2, \dots unabhängige i.i.d.-neg. heltallige } alle unabh.

$$N \sim \mu(\lambda)$$

$$S_N = \sum_{k=1}^N X_k \sim \mu(\mu)$$

$$\begin{aligned}
 G_{S_N}(s) &= (G_N \circ G_X)(s) = e^{\mu(s-1)} = e^{\lambda\left(\frac{\mu}{\lambda}s - 1\right)} \\
 &= e^{\lambda\left(1 - \frac{\mu}{\lambda} + \frac{\mu}{\lambda}s - 1\right)}
 \end{aligned}$$

$$\Rightarrow G_X(s) = 1 - \frac{\mu}{\lambda} + \frac{\mu}{\lambda}s \Rightarrow X \sim \mathcal{U}\left(1, \frac{\mu}{\lambda}\right)$$



138

X : $Y = 4X$ er heltallig

$$N \sim \mu(20) \quad X, N \text{ unabh.}$$

$$E[4S_N] = EN EY = 20 \cdot \left(3 \frac{1}{4} + 4 \frac{1}{4}\right) = 20 \frac{7}{4} = 35$$

$$E S_N = \frac{1}{4} 35 = 8,75 \text{ \$}$$

140

$X_k \sim \mathcal{U}\{1, 2, 3, 4\}$, $k=1, \dots, N$ } alle unabh.

$$N \sim \mu(4)$$

$$Y = \sum_{k=1}^N X_k$$

$$a \quad G_X(s) = s \frac{1}{4} + s^2 \frac{1}{4} + s^3 \frac{1}{4} + s^4 \frac{1}{4} = \frac{1}{4} (s + s^2 + s^3 + s^4)$$

$$G_N(s) = e^{4(s-1)}$$

$$G_Y(s) = (G_N \circ G_X)(s) = \exp(s + s^2 + s^3 + s^4 - 4)$$

fortsettes

$$b \quad P(Y=0) = G_Y(0) = e^{-4}$$

$$c \quad EY = EN \cdot EX = 4 \cdot \frac{5}{2} = 10$$

$$\begin{aligned} \text{Var } Y &= EN \cdot \text{Var } X + \text{Var } N (EX)^2 \\ &= EN (E[X^2] - (EX)^2) + \text{Var } N (EX)^2 \\ &= EN \cdot E[X^2] + (\text{Var } N - EN) (EX)^2 \\ &= 4 \cdot \frac{4 \cdot 5 \cdot 9}{6} \cdot \frac{1}{4} + (4 - 4) (EX)^2 \\ &= 30 \end{aligned}$$

142 X_1, X_2, \dots iku-neg. ensfordelte m. ford. fkt. F } alle uafh.
 N iku-neg. heltallig m. sands. fr. br. fkt. G }

$$M_N = \text{matrix } \{X_1, X_2, \dots, X_N\}, \quad M_N = 0 \text{ for } N=0$$

$$\begin{aligned} a \quad F_{M_N}(x) &= P(M_N \leq x) = \sum_{n=0}^{\infty} P(M_N \leq x | N=n) P(N=n) \\ &= \sum_{n=0}^{\infty} (F(x))^n P(N=n) = (G \circ F)(x) \end{aligned}$$

M_N kont., når X kont.

$$b \quad X \sim U[0; 100], \text{ des. } F(x) = \frac{x}{100}, \quad 0 \leq x \leq 100$$

$$N \sim \mu(2)$$

$$F_{M_N}(x) = e^{2\left(\frac{x}{100} - 1\right)} \geq 0,9 \Rightarrow x \geq 94,73$$

147 X_1, X_2, \dots ensfordelte m. moment fr. br. fkt. M_X } alle uafh.
 N iku-neg. heltallig m. sands. fr. br. fkt. G_N }

$$\begin{aligned} a \quad M_{S_N}(t) &= E[e^{tS_N}] = \sum_{n=0}^{\infty} E[e^{tS_N} | N=n] P(N=n) \\ &= \sum_{n=0}^{\infty} E[e^{tS_n}] P(N=n) = \sum_{n=0}^{\infty} (M_X(t))^n P(N=n) \\ &= (G_N \circ M_X)(t) \end{aligned}$$

$$b \quad X_k \sim e(\lambda), \quad N \sim g(\mu)$$

$$M_{S_N}(t) = \frac{\mu \frac{\lambda}{\lambda-t}}{1 - \frac{\lambda}{\lambda-t} + \mu \frac{\lambda}{\lambda-t}} = \frac{\lambda \mu}{\lambda \mu - t} \Rightarrow S_N \sim e(\lambda \mu)$$

fortsattes

b fortsat

S_N kan opfattes som ventetid ind. begivenheder
i en udtyndet Poissonprocess. (opg. 3.12 153)

$$\begin{aligned} c \quad E S_N &= (G_N' \circ M_X)(0) \cdot M_X'(0) = G_N'(1) \cdot M_X'(0) \\ &= EN \cdot EX \end{aligned}$$

$$149 \quad G(s,t) = E[s^X t^Y], \quad X, Y \text{ uafh. uafh. heltallige} \\ 0 \leq s, t \leq 1$$

$$a \quad G_X(s) = G(s, 1), \quad G_Y(t) = G(1, t)$$

$$b \quad EX = G_s(1, 1), \quad EY = G_t(1, 1)$$

$$\text{Var } X = G_{s^2}(1, 1) + G_s(1, 1) - (G_s(1, 1))^2$$

$$\text{Var } Y = G_{t^2}(1, 1) + G_t(1, 1) - (G_t(1, 1))^2$$

$$\text{Cov}[X, Y] = G_{st}(1, 1) - G_s(1, 1) G_t(1, 1)$$

$$c \quad i \quad G(s, t) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} s^j t^k r_{jk}, \quad r_{00} = G(0, 0)$$

$$G_{s^j t^k}(0, 0) = j! k! r_{jk} \Rightarrow r_{jk} = \frac{G_{s^j t^k}(0, 0)}{j! k!}$$

der. r_{jk} entydigt bestemt af G for alle (j, k)

$$ii \quad X, Y \text{ uafh.} \Rightarrow G(s, t) = E[s^X] E[t^Y] \\ = G_X(s) G_Y(t)$$

$$G(s, t) = G_X(s) G_Y(t) \Rightarrow G_{s^j t^k}(0, 0) = G_X^{(j)}(0) G_Y^{(k)}(0)$$

$$\Rightarrow j! k! r_{jk} = j! r_j k! r_k \Rightarrow r_{jk} = r_j r_k$$

$$\Rightarrow X, Y \text{ uafh.}$$

$$d \quad i \quad M(s, t) = E[e^{sX} e^{tY}]$$

$$ii \quad M_X(s) = M(s, 0), \quad M_Y(t) = M(0, t)$$

$$iii \quad EX = M_s(0, 0), \quad \text{Var } X = M_{s^2}(0, 0) - (M_s(0, 0))^2$$

$$EY = M_t(0, 0), \quad \text{Var } Y = M_{t^2}(0, 0) - (M_t(0, 0))^2$$

$$\text{Cov}[X, Y] = M_{st}(0, 0) - M_s(0, 0) M_t(0, 0)$$

150 Poissonproces m. intensitet $\lambda = \frac{1}{13}$ pr. år

a $t = 10$, $X \sim \mu\left(\frac{10}{13}\right)$

$$P(X=0) = e^{-\frac{10}{13}} = 0,4634$$

b $T \sim e\left(\frac{1}{13}\right)$

$$P(T > 5) = e^{-\frac{5}{13}} = 0,6807$$

c $t = 13$, $Y \sim \mu\left(\frac{13}{13}\right) = \mu(1)$

$$P(Y=1) = e^{-1} = 0,3679$$

d $(P(X=1))^3 = \left(\frac{10}{13} e^{-\frac{10}{13}}\right)^3 = 0,0453$

151 Poissonproces m. intensitet $\lambda = 2$ pr. uge

a $t = \frac{1}{7}$, $X \sim \mu\left(\frac{2}{7}\right)$

$$P(X=0) = e^{-\frac{2}{7}} = 0,7515$$

b $P(\text{personskade}) = \frac{1}{10}$

udtyndet Poissonproces m. intensitet $\lambda p = 2 \frac{1}{10} = \frac{1}{5}$

$$t = \frac{30,44}{7} \quad (\text{en ges. måned sat til } 30,44 \text{ dg.})$$

$$Y \sim \mu(\lambda p t) = \mu\left(\frac{30,44}{35}\right)$$

$$P(Y \geq 1) = 1 - P(Y=0) = 1 - e^{-\frac{30,44}{35}} = 0,5809$$

c N : antal uheldsfor uret pr. år, $Z \sim \mu(2)$

$$N \sim G(52, P(Z=0)) = G(52, e^{-2})$$

*

157 $X \sim \mu(2t)$, $Y \sim \mu(3t)$ u.o/h., $X+Y \sim \mu(5t)$

a $t=1$, $P(X+Y=2) = \frac{e^{-5} 5^2}{2!} = 0,042$

b $t=1$, $P(X=1, Y=1) = P(X=1)P(Y=1) = 2e^{-2} 3e^{-3} = 0,0404$

c $t=1$, $P(Y=2 | X+Y=2) = \binom{2}{2} \left(\frac{3}{5}\right)^2 \left(\frac{2}{5}\right)^0 = \left(\frac{3}{5}\right)^2 = \frac{9}{25} = 0,36$

d $t=10$, $Z_1, \dots, Z_4 \sim U[0; 10]$ u.o/h.

$$\begin{aligned} \binom{4}{2} P(Z_j \leq 5, Z_k \leq 5) &= \binom{4}{2} P(Z_j \leq 5) P(Z_k \leq 5), \quad j \neq k \\ &= 6 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = \frac{3}{8} = 0,375 \end{aligned}$$

* Løsning til opgave 156, se nederst næste side

Poissonproces m. intensitet $\lambda = 2$ pr. uge, $X \sim p(2)$

$$a \quad P(Y=k) = \frac{e^{-2} 2^{2k-1}}{(2k-1)!}, \quad k=1, 2, \dots$$

svare ikke til en Poissonproces

$$b \quad P(X \leq 1) = P(X=0) + P(X=1) = e^{-2} + 2e^{-2} = 3e^{-2} = 0,4060$$

$$c \quad P(Z=k) = \frac{e^{-2} 2^{2k}}{(2k)!}, \quad k=0, 1, \dots$$

$$\sum_{k=0}^{\infty} P(Z=k) = e^{-2} \sum_{k=0}^{\infty} \frac{2^{2k}}{(2k)!} = e^{-2} \cosh 2 = \frac{1+e^{-4}}{2} = 0,5092$$

To Poissonprocesser m. intensitet μ og λ

X : antal begivenheder i første proces mellem to konsekutive begivenheder i anden proces

T : ventetid ind. to konsekutive begivenheder i anden proces.

$$P(X=k(T)) = \int_0^{\infty} P(X=k) f_T(t) dt, \quad \text{H. satn. 3.5.1 (b) side 170}$$

$$= \int_0^{\infty} \frac{e^{-\mu t} (\mu t)^k}{k!} \lambda e^{-\lambda t} dt$$

$$= \frac{\mu^k \lambda}{(\mu + \lambda)^{k+1}} \int_0^{\infty} \frac{(\mu + \lambda)^{k+1}}{\Gamma(k+1)} t^{(k+1)-1} e^{-(\mu + \lambda)t} dt$$

$$= \left(\frac{\mu}{\mu + \lambda} \right)^k \frac{\lambda}{\mu + \lambda}, \quad k=0, 1, \dots$$

$$\Rightarrow P(Y=j(T)) = \left(\frac{\mu}{\mu + \lambda} \right)^{j-1} \frac{\lambda}{\mu + \lambda}, \quad j=1, 2, \dots$$

$$Y \sim g\left(\frac{\lambda}{\mu + \lambda}\right)$$

$$P(X=k) = \binom{n}{k} \left(\frac{\lambda_1 t}{\lambda_1 t + \lambda_2 t} \right)^k \left(\frac{\lambda_2 t}{\lambda_1 t + \lambda_2 t} \right)^{n-k}, \quad k=0, 1, \dots, n$$