

4.2

$$1 \quad X_1, X_2, \dots, E X_n = \mu, \text{Var } X_k = \sigma^2, k=1,2,\dots$$

$$\text{Cov}(X_j, X_k) < 0, j \neq k$$

$$\bar{X} = \frac{1}{n} \sum_{k=1}^n X_k$$

$$\text{Var } \bar{X} = \frac{1}{n^2} \left(\sum_k \text{Var } X_k + \sum_{j \neq k} \text{Cov}(X_j, X_k) \right) < \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$$P(|\bar{X} - \mu| > \varepsilon) \leq \frac{\text{Var } \bar{X}}{\varepsilon^2} < \frac{\sigma^2}{n\varepsilon^2} \rightarrow 0 \text{ for } n \rightarrow \infty$$

$$\Leftrightarrow \bar{X} \xrightarrow{P} \mu$$

$$2 \quad X_1, X_2, \dots \quad X_n \xrightarrow{P} a \text{ dws. } P(|X_n - a| > \varepsilon) \rightarrow 0 \text{ for } n \rightarrow \infty$$

$$Y_1, Y_2, \dots \quad Y_n \xrightarrow{P} b \text{ dws. } P(|Y_n - b| > \varepsilon) \rightarrow 0 \text{ for } n \rightarrow \infty$$

$$P(|X_n + Y_n - (a+b)| > \varepsilon) = P(|X_n - a + Y_n - b| > \varepsilon)$$

$$\leq P(|X_n - a| + |Y_n - b| > \varepsilon)$$

$$\leq P(|X_n - a| > \frac{\varepsilon}{2}) + P(|Y_n - b| > \frac{\varepsilon}{2}) \rightarrow 0$$

for $n \rightarrow \infty$

$$\Rightarrow X_n + Y_n \xrightarrow{P} a + b$$

$$3 \quad X_1, X_2, \dots \text{ unif. }, X_k \sim U[0,1], k=1,2,\dots$$

$g : [0,1] \rightarrow \mathbb{R}$ reell fkt.

$$\frac{1}{n} \sum_{k=1}^n g(X_k) \xrightarrow{P} E[g(X_k)] = \int_0^1 g(x) \cdot 1 dx = \int_0^1 g(x) dx$$

$$4 \quad X_1, \dots, X_n \text{ unif., einfordelte m. tath. } f(x) = 3x^2, 0 \leq x \leq 1$$

$$H_n = \left(\frac{1}{n} \sum_{k=1}^n \frac{1}{X_k} \right)^{-1} \Leftrightarrow H_n^{-1} = \frac{1}{n} \sum_{k=1}^n Y_k, Y_k = \frac{1}{X_k}$$

$$y = \frac{1}{x} \Leftrightarrow x = \frac{1}{y} \Rightarrow \frac{dx}{dy} = -\frac{1}{y^2}$$

$$f_Y(y) = f\left(\frac{1}{y}\right) \left| -\frac{1}{y^2} \right| = 3y^{-4}, 1 \leq y < \infty$$

$$E Y_k = \int_1^\infty y \cdot 3y^{-4} dy = \left[-\frac{3}{2} y^{-2} \right]_1^\infty = \frac{3}{2}$$

$$H_n^{-1} \xrightarrow{P} \frac{3}{2} \Rightarrow H_n \xrightarrow{P} \frac{2}{3} \text{ jf. Noveller 4.2 s. 267}$$

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math., ensförderte m. tech. $f(x) > 0$,
 $0 \leq x \leq a$
 ferd. fkt. F er vaks. fkt. af x , $0 \leq x \leq a$

$$P(|X_{(1)} - 0| > \varepsilon) = P(X_{(1)} > \varepsilon) = (1 - F(\varepsilon))^n \rightarrow 0 \quad \text{for } n \rightarrow \infty$$

$$\Leftrightarrow X_{(1)} \xrightarrow{P} 0$$

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a $X \sim \Gamma(n, \lambda)$, $EX = \frac{n}{\lambda}$, $\text{Var } X = \frac{n}{\lambda^2}$
 $X = \sum_{k=1}^n Y_k$, $Y_k \sim \text{exp}(\lambda)$, $k=1, \dots, n$, math.

$$X \sim N\left(\frac{n}{\lambda}, \frac{n}{\lambda^2}\right) \text{ app.}$$

b $X \sim n(\lambda)$, λ 'stor', $EX = \lambda$, $\text{Var } X = \lambda$

$X = \sum_{k=1}^n Y_k$, $Y_k \sim n\left(\frac{\lambda}{m}\right)$, $k=1, \dots, m$, math.
 $X \sim N(\lambda, \lambda)$ app.

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$X \sim U\{1, \dots, 6\}$, $EX = 3,5$, $\text{Var } X = \frac{35}{12}$

$$\bar{X} \sim N\left(3,5; \frac{35}{12m}\right) \text{ app.}$$

$$P(3 < \bar{X} < 4) = P(|\bar{X} - 3,5| < 0,5) = 2 \Phi\left(\frac{0,5}{\sqrt{\frac{35}{12m}}}\right) - 1 > 0,99$$

$$\Rightarrow \Phi\left(\frac{0,5}{\sqrt{\frac{35}{12m}}}\right) > 0,995 \Rightarrow \frac{0,5}{\sqrt{\frac{35}{12m}}} > 2,576 \Rightarrow m > 77,42$$

$$m \geq 78$$

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multiple choice, 100 spørsmål m. hva. 4 svarenumheder

$$P(\text{richtigt svar}) = 1 \cdot \frac{3}{4} + \frac{1}{4} \cdot \frac{1}{4} = \frac{13}{16} \quad (P(\text{viden}) = \frac{3}{4})$$

X : antal rigtige svar, $X \sim b(100, \frac{13}{16}, 100 \cdot \frac{13}{16} \cdot \frac{3}{16})$

$$X \sim N\left(\frac{325}{4}, \frac{975}{64}\right) \text{ app.}$$

$$P(\text{vestå}) = P(X \geq 80) \approx \Phi\left(-\frac{79,5 - 81,25}{\frac{5}{8}\sqrt{39}}\right) = \Phi(0,44836)$$

$$= 0,6857 \quad (\text{kontinuitetskorraktion anvendt})$$

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15 200 kg, antal kilo
pr. kg.

$$\mu = 0,6 + 2 \cdot 0,3 = 1,2$$

$$\sigma^2 = 0,6 + 4 \cdot 0,3 - (1,2)^2$$

$$= 1,8 - 1,44 = 0,36$$

X: antal kilo

$$X \sim N(200 \cdot 1,2, 200 \cdot 0,36) = N(240, 72) \text{ a.s.m.}$$

$$\Phi\left(\frac{x-240}{\sqrt{72}}\right) > 0,95 \Rightarrow \frac{x-240}{\sqrt{72}} > 1,645 \Rightarrow x > 253,96$$

$$x \geq 254$$

16 i $EY = \mu$, $\text{Var} Y = \sigma^2$, g diff. flt.

$$g(X) = g(\mu) + g'(\mu)(X-\mu) + \begin{cases} R_1 \\ \frac{1}{2}g''(\mu)(X-\mu)^2 + R_2 \end{cases}$$

$$E[g(X)] \approx \begin{cases} g(\mu) + 0 = g(\mu) \\ g(\mu) + 0 + \frac{1}{2}g''(\mu)\sigma^2 = g(\mu) + \frac{1}{2}\sigma^2 g''(\mu) \end{cases}$$

ii $X \sim U[0;1]$, $\mu = \frac{1}{2}$, $\sigma^2 = \frac{1}{12}$

a $g(x) = x$, $EY \approx \begin{cases} \frac{1}{2} \\ \frac{1}{2} + 0 = \frac{1}{2} \end{cases}$

b $g(x) = x^2$, $EY \approx \begin{cases} \frac{1}{4} \\ \frac{1}{4} + \frac{1}{2} \frac{1}{12} \cdot 2 = \frac{3+1}{12} = \frac{1}{3} \end{cases}$

c $g(x) = x^4$, $EY \approx \begin{cases} \left(\frac{1}{2}\right)^4 \\ \left(\frac{1}{2}\right)^4 + \frac{1}{2} \frac{1}{12} 4 \cdot 3 \left(\frac{1}{2}\right)^2 = \frac{3}{16} \end{cases}$

d $g(x) = e^x$, $EY \approx \begin{cases} e^{\frac{1}{2}} = 1,6487 \\ e^{\frac{1}{2}} + \frac{1}{2} \frac{1}{12} e^{\frac{1}{2}} = \frac{25}{24} e^{\frac{1}{2}} = 1,7174 \end{cases}$

e $g(x) = e^{-x}$, $EY \approx \begin{cases} e^{-\frac{1}{2}} = 0,6065 \\ e^{-\frac{1}{2}} + \frac{1}{2} \frac{1}{12} e^{-\frac{1}{2}} = \frac{25}{24} e^{-\frac{1}{2}} = 0,6318 \end{cases}$

f $g(x) = \sin(\pi x)$, $EY \approx \begin{cases} \sin \frac{\pi}{2} = 1 \\ \sin \frac{\pi}{2} + \frac{1}{2} \frac{1}{12} \pi^2 \sin \frac{\pi}{2} = 1 - \frac{\pi^2}{24} = 0,5888 \end{cases}$

g $g(x) = \sin(2\pi x)$, $EY \approx \begin{cases} \sin \pi = 0 \\ \sin \pi + \frac{1}{2} \frac{1}{12} \pi^2 \sin \pi = 0 + 0 = 0 \end{cases}$

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avg. 4.2 4. fortset

$$EY_k^2 = \int_1^\infty y^2 \cdot 3y^{-4} dy = [-3y^{-1}]_1^\infty = 3$$

$$\text{Var } Y_k = 3 - (\frac{3}{2})^2 = \frac{3}{4}$$

$$\bar{Y} \sim N(\frac{3}{2}, \frac{3}{4}) \text{ app.}$$

$$H_m = \bar{Y}^{-1} \sim N\left((\frac{3}{2})^{-1}, \frac{\frac{3}{4}(-\frac{1}{(\frac{3}{2})^2})}{m}\right) = N\left(\frac{2}{3}, \frac{4}{27m}\right) \text{ app.}$$

H. satz 4.1 s. 273

4.4

19 $X \sim h(N, r, n)$

$$P(X=k) = \frac{\binom{r}{k} \binom{N-r}{n-k}}{\binom{N}{n}} = \frac{(r)_k (N-r)_{n-k} n!}{k! (n-k)! (N)_n}$$

$$= \frac{\frac{(N-r)_n}{(N-k)_n} \frac{(n-r)_k}{(n-r-n+k)_k} \frac{(n)_k r^k}{(N)_k} \frac{(r)_k}{r^k}}{k!} \rightarrow \frac{e^{-\lambda} \cdot 1 \cdot \lambda^k \cdot 1}{k!} \text{ for } \begin{cases} r \rightarrow \infty \\ m \rightarrow \infty \\ N \rightarrow \infty \\ \frac{r}{N} \rightarrow 0 \\ \frac{n}{N} \rightarrow 0 \\ \frac{rn}{N} = \lambda \end{cases}$$

d.h. $X \xrightarrow{D} Y \sim p(\lambda)$

Desuden

$$P(X=k) = \binom{m}{k} \frac{(r)_k}{(N)_n} \frac{(N-r)_{m-k}}{(N-k)_{m-k}} \rightarrow \binom{m}{k} p^k (1-p)^{m-k}$$

d.h. $X \xrightarrow{D} Y \sim b(m, p)$

for $\begin{cases} r \rightarrow \infty \\ N \rightarrow \infty \\ \frac{r}{N} \rightarrow p \\ \frac{m}{N} = p \end{cases}$ (konst.)

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$x_1, x_2, \dots, x_n \sim U\{1, \dots, 6\}$; $X \sim U[0; 1]$

$$P\left(\frac{1}{m} X_m \leq x\right) = P(X_m \leq mx) = \frac{[mx]}{m} \rightarrow x \text{ for } m \rightarrow \infty,$$

$0 \leq x \leq 1$

$\Leftrightarrow F_{\frac{1}{m} X_m}(x) \rightarrow F_X(x) \text{ for } m \rightarrow \infty$

$\Leftrightarrow \frac{1}{m} X_m \xrightarrow{D} X$

21 $X_1, X_2, \dots, X_n \sim g\left(\frac{1}{n}\right) ; X \sim e(1)$

$$\begin{aligned} P\left(\frac{1}{n}X_n \leq x\right) &= P(X_n \leq nx) = \sum_{k=1}^{[nx]} \frac{1}{n} \left(1 - \frac{1}{n}\right)^{k-1} \\ &= \frac{1}{n} \left(1 - \frac{1}{n}\right) \frac{1 - (1 - \frac{1}{n})^{[nx]}}{1 - (1 - \frac{1}{n})} = \left(1 - \frac{1}{n}\right) \left(1 - \left(1 - \frac{1}{n}\right)^{[nx]}\right), \quad [nx] = m \\ &\approx \left(1 - \frac{x}{m}\right) \left(1 - \left(1 - \frac{x}{m}\right)^m\right) \rightarrow 1 - e^{-x} \text{ for } m \rightarrow \infty, 0 \leq x < \infty \\ \Leftrightarrow F_{\frac{1}{n}X_n}(x) &\rightarrow F_X(x) \text{ for } n \rightarrow \infty \Leftrightarrow \frac{1}{n}X_n \xrightarrow{D} X \end{aligned}$$

22 X_1, X_2, \dots unif., $X_n \sim U[0; 1]$

$$\begin{aligned} F_{X_{(n)}}(y) &= (F_{X_{(1)}}(x))^n = x^n ; Y_n = n(1 - X_{(n)}), Y \sim e(1) \\ P(Y_n \leq y) &= P(n(1 - X_{(n)}) \leq y) = P(X_{(n)} \geq 1 - \frac{y}{n}) \\ &= 1 - (1 - \frac{y}{n})^n \rightarrow 1 - e^{-y} \text{ for } n \rightarrow \infty, 0 \leq y < \infty \\ \Leftrightarrow F_{Y_n}(y) &\rightarrow F_Y(y) \text{ for } n \rightarrow \infty \Leftrightarrow Y_n \xrightarrow{D} Y \end{aligned}$$

24 X_1, X_2, \dots unif., ausforderte nn. teeth. $f(x) = 2x, 0 \leq x \leq 1$

$$F_{X_k}(x) = \int_0^x 2v dv = x^2, 0 \leq x \leq 1$$

$$Y_n = \sqrt{n}X_{(1)}, F_{X_{(1)}}(x) = 1 - (1 - x^2)^n, 0 \leq x \leq 1$$

$$\begin{aligned} F_{Y_n}(y) &= P(Y_n \leq y) = P(\sqrt{n}X_{(1)} \leq y) = P(X_{(1)} \leq \frac{y}{\sqrt{n}}) \\ &= 1 - (1 - (\frac{y}{\sqrt{n}})^2)^n = 1 - (1 - \frac{y^2}{n})^n \rightarrow 1 - e^{-y^2} \text{ for } n \rightarrow \infty, \\ &\Leftrightarrow Y_n \xrightarrow{D} Y \sim W(2, 1) \end{aligned}$$

25 X_1, X_2, \dots unif., ausforderte nn. teeth. $f(x) = 3x^2, 0 \leq x \leq 1$

$$F_{X_k}(x) = \int_0^x 3x^2 dx = x^3, 0 \leq x \leq 1$$

$$Y_n = a_n X_{(1)}, F_{X_{(1)}}(x) = 1 - (1 - x^3)^n, 0 \leq x \leq 1$$

$$\begin{aligned} F_{Y_n}(y) &= P(Y_n \leq y) = P(a_n X_{(1)} \leq y) = P(X_{(1)} \leq \frac{y}{a_n}) \\ &= 1 - (1 - (\frac{y}{a_n})^3)^n = 1 - (1 - \frac{y^3}{a_n^3})^n \end{aligned}$$

$$\begin{aligned} a_n &= \sqrt[3]{n} \Rightarrow F_{Y_n}(y) = 1 - (1 - \frac{y^3}{n})^n \rightarrow 1 - e^{-y^3} \text{ for } n \rightarrow \infty, \\ &\Leftrightarrow Y_n \xrightarrow{D} Y \sim W(3, 1) \end{aligned}$$