

1 H. opg. 2.4 25 a

$$X = \begin{cases} 0, & \text{mås } 0 \leq u \leq \frac{1}{8} \\ 1, & \text{mås } \frac{1}{8} < u \leq \frac{1}{2} \\ 2, & \text{mås } \frac{1}{2} < u \leq \frac{3}{8} \\ 3, & \text{mås } \frac{3}{8} < u \leq 1 \end{cases}, \quad U \sim U[0;1]$$

2 H. opg. 2.4 25 b

$$X = \begin{cases} 0, & \text{mås } 0 \leq u \leq \frac{1}{4} \\ 1, & \text{mås } \frac{1}{4} < u \leq \frac{3}{4}, \quad U \sim U[0;1] \\ 2, & \text{mås } \frac{3}{4} < u \leq 1 \end{cases}$$

3  $Y \sim n(\lambda) = p(1, \lambda)$ , dvs.  $Y$  kan opfattes som antal begivenheder i  $[0; \lambda]$  i en poissonproces m. intensitet 1. $S_m$ : tid til n'te begivenhed

$$S_m = \sum_{k=1}^m X_k, \quad X_k \sim e(1), \quad k=1, \dots, m, \quad \text{uafh.}$$

$$S_m = -\sum_{k=1}^m \ln U_k, \quad U_k \sim U[0;1], \quad k=1, \dots, m, \quad \text{uafh.}$$

$$\begin{aligned} Y &= \min \{ n \mid S_{n+1} > \lambda \} = \min \{ n \mid S_n > \lambda \} + 1 \\ &= \min \{ n \} - \sum_{k=1}^m \ln U_k > \lambda \} + 1 \end{aligned}$$

4  $N \sim n(\lambda t)$ 

$$N = \min \{ n \} - \sum_{k=1}^m \ln U_k > \lambda t \} + 1, \quad \text{H. opg. 3 med } \lambda := \lambda t$$



$$s_j = t U_{(j)}, \quad j=1, \dots, n, \quad U_{(j)} \sim U[0;1], \quad j=1, \dots, n, \quad \text{uafh.}$$

jf. 'order statistic property'

5  $X \sim \text{nb}(r, p)$ 

$$X = \sum_{j=1}^r Y_j, \quad Y_j \sim g(p), \quad j=1, \dots, r, \quad \text{uafh.}$$

$$P(Y_j = k) = P(Y_j \leq k) - P(Y_j \leq k-1) = P(Y_j > k-1) - P(Y_j > k)$$

$$Y_j = k, \quad \text{mås } (1-p)^k < u < (1-p)^{k-1}, \quad U \sim U[0;1]$$

Bemerk, at  $(1-p)^k < u < (1-p)^{k-1} \Leftrightarrow \frac{\ln u}{\ln(1-p)} < k < \frac{\ln u}{\ln(1-p)} + 1$ ,

$$\text{dvs. } Y_j = \left[ \frac{\ln U_j}{\ln(1-p)} \right] + 1, \quad U \sim U[0;1].$$

$$X = \sum_{j=1}^r \left[ \frac{\ln U_j}{\ln(1-p)} \right] + r, \quad U_j \sim U[0;1], \quad j=1, \dots, r$$

uafh.

7  $X$  har tæth.  $f(x) = 3x^2$ ,  $0 \leq x \leq 1$

$$F(x) = \int_0^x 3t^2 dt = [t^3]_0^x = x^3, 0 \leq x \leq 1$$

$$u = x^3 \Leftrightarrow x = u^{\frac{1}{3}}$$

$$X = u^{\frac{1}{3}}, u \sim U[0; 1]$$

8  $X \sim \text{Cauchy}$ , tæth.  $f(x) = \frac{1}{\pi(1+x^2)}$ ,  $-\infty < x < \infty$

$$\text{a} \quad F(x) = \int_{-\infty}^x \frac{1}{\pi(1+t^2)} dt = \frac{1}{\pi} [\arctg t]_{-\infty}^x = \frac{1}{\pi} \arctg x + \frac{1}{2}$$

$$u = \arctg x + \frac{1}{2} \Leftrightarrow x = \operatorname{tg}(\pi(u - \frac{1}{2}))$$

$$X = \operatorname{tg}(\pi(u - \frac{1}{2})), u \sim U[0; 1]$$

b  $X, Y \sim \text{Cauchy}$  mafh.

$$Z = X+Y$$

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)} \frac{1}{\pi(1+(z-x)^2)} dx$$

$$= \frac{1}{\pi^2} \lim_{r \rightarrow \infty} \oint_C \frac{1}{1+w^2} \frac{1}{1+(z-w)^2} dw$$



singulære punkter:  
 $w = \pm i$  og  $w = z \pm i$

$$= \frac{1}{\pi^2} 2\pi i \left( \frac{1}{2i} \frac{1}{1+(z-i)^2} + \frac{1}{-2i} \frac{1}{1+(z+i)^2} \right)$$

$$= \frac{i}{\pi} \left( \frac{1}{z^2-2iz} + \frac{1}{z^2+2iz} \right)$$

$$= \frac{1}{\pi z} \frac{z+2i+z-2i}{z^2+4} = \frac{2}{\pi(4+z^2)}$$

$$f_{\frac{X+Y}{2}}(z) = \frac{2}{\pi(4+(2z)^2)} |2| = \frac{1}{\pi(1+z^2)}$$

$$\text{dvs. } \frac{X+Y}{2} \sim \text{Cauchy}$$

Hvis af ses, at med  $X_1, \dots, X_n \sim \text{Cauchy}$  mafh.,

bliver  $\bar{X} \sim \text{Cauchy}$  ( $n = 2^m$ ).

I Cauchy-fordelingen givs det altid ikke  
fordelt nogensteds at tage gennemsnit af  
flere observationer. ( $\bar{X} \sim \text{Cauchy}$ ,  $n \neq 2^m$ , vises ved  
brugtelse af karakteristisk fkt.).

9  $X \sim W(\alpha, \lambda)$

$$F(x) = 1 - e^{-\lambda x^\alpha}, \quad 0 \leq x < \infty$$

$$u = 1 - e^{-\lambda x^\alpha} \Leftrightarrow x^\alpha = -\frac{1}{\lambda} \ln(1-u) \Rightarrow x = \left(-\frac{1}{\lambda} \ln(1-u)\right)^{\frac{1}{\alpha}}$$

$$X = \left(-\frac{1}{\lambda} \ln U\right)^{\frac{1}{\alpha}}, \quad U \sim U[0;1] \quad (U \sim 1-U)$$

10  $X \sim g(r)$

Fra opg. 2.6 66:  $X = [T] + 1, \quad T \sim e(n), \quad r = 1 - e^{-n}$

$$T = -\frac{1}{n} \ln U = \frac{\ln U}{\ln e^{-n}} = \frac{\ln U}{\ln(1-r)}, \quad U \sim U[0;1]$$

$$X = \left[ \frac{\ln U}{\ln(1-r)} \right] + 1$$

Samme resultat som mellemresultatet i opg. 5.3 5.

11  $P(X=0) = 0,2$

$$F_x(x) = 1 - 0,8 e^{-x}, \quad 0 \leq x < \infty$$

$$u = 1 - 0,8 e^{-x} \Leftrightarrow x = -\ln \frac{5(1-u)}{4}$$

$$X = \begin{cases} 0, & \text{når } 0,8 < u \leq 1 \\ -\ln \frac{5u}{4}, & \text{når } 0 < u \leq 0,8 \end{cases}, \quad U \sim U[0;1]$$

13  $X \sim e(1), \quad X = -\ln U, \quad U \sim U[0;1]$

$$\ln X = \ln(-\ln U), \quad U \sim U[0;1]$$

$$E[\ln X] \approx \frac{1}{m} \sum_{k=1}^m \ln(-\ln u_k), \quad u_k \sim U[0,1], \\ k = 1, \dots, m, \quad \text{uafh.}$$

14  $X \sim N(0,1), \quad \text{tæth. } \varphi(x), \quad Y = \sin X$

$$\int_{-\infty}^{\infty} |\sin x| \varphi(x) dx \leq \int_{-\infty}^{\infty} 1 \varphi(x) dx = 1 \Rightarrow E[\sin X] \text{ eks.}$$

$$E[\sin X] = \int_{-\infty}^{\infty} \sin x \varphi(x) dx = 0$$

simulering ikke nødvendig

15  $X, Y, Z \sim U[0;1] \quad \text{uafh.},$

$$P(X+Y+Z \leq 2,5) \approx \frac{1}{m} \sum_{k=1}^m I_k, \quad I_k = \begin{cases} 1, & \text{når } U_1 + U_2 + U_3 \leq 2,5 \\ 0, & \text{når } U_1 + U_2 + U_3 > 2,5 \end{cases}, \\ U_1, U_2, U_3 \sim U[0;1] \quad \text{uafh.}$$

16  $g: [0;1] \rightarrow \mathbb{R}, \quad I = \int_0^1 g(x) dx$

$$E[g(U)] = \int_0^1 g(u) \cdot 1 du = I, \quad u \sim U[0;1]$$

$$I \approx \frac{1}{n} \sum_{k=1}^n g(u_k), \quad u_k \sim U[0;1], \quad k=1, \dots, n, \quad \text{wah.}$$

17 Box-Muller:  $X = \sqrt{-2 \ln U} \cos(2\pi V)$ ,  $Y = \sqrt{-2 \ln U} \sin(2\pi V)$ ,  $U, V \sim U[0;1]$   
wah.

$$\text{Transformation: } X_1 = \sigma_1 X + \mu_1$$

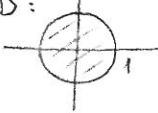
$$Y_1 | X_1 = \sigma_2 \sqrt{1-\rho^2} Y + \mu_2 | X_1$$

$$= \sigma_2 \sqrt{1-\rho^2} Y + \mu_2 + \rho \frac{\sigma_2}{\sigma_1} (X_1 - \mu_1)$$

$$(X_1, Y_1) = (\sigma_1 X + \mu_1, \sigma_2 (\rho X + \sqrt{1-\rho^2} Y) + \mu_2)$$

$$(X_1, Y_1) \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$$

18

$D:$    $f(r, \theta) = \frac{1}{\pi}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta < 2\pi, \quad x = r \cos \theta, \quad y = r \sin \theta$

$$f_R(r) = \int_0^{2\pi} \frac{1}{\pi} r d\theta = \frac{r}{\pi} [\theta]_0^{2\pi} = 2r$$

$$F_R(r) = \int_0^r 2t dt = 2 \left[ \frac{t^2}{2} \right]_0^r = r^2, \quad u = r^2 \Rightarrow r = \sqrt{u}$$

$$f_\theta(\theta) = \int_0^1 \frac{1}{\pi} r dr = \frac{1}{\pi} \left[ \frac{r^2}{2} \right]_0^1 = \frac{1}{2\pi} \Rightarrow \theta \sim U[0; 2\pi]$$

$$(X, Y) = (\sqrt{u} \cos(2\pi v), \sqrt{u} \sin(2\pi v)),$$

$$U, V \sim U[0;1] \quad \text{wah}$$

$$(X, Y) \sim U[D]$$

19  $X, \text{ teeth. } f(x) = \frac{4}{\sqrt{\pi}} x^2 e^{-x^2}, \quad 0 \leq x < \infty$

$$Y \sim \Gamma(3, 1), \quad g(y) = \frac{1}{2} y^2 e^{-y}$$

$$\begin{aligned} \frac{f(y)}{c g(y)} &= \frac{\frac{4}{\sqrt{\pi}} y^2 e^{-y^2}}{c \frac{1}{2} y^2 e^{-y}} = \frac{1}{c} \frac{8}{\sqrt{\pi}} e^{-(y^2-y+\frac{1}{4})} e^{\frac{1}{4}} \\ &= \frac{1}{c} \frac{8 e^{\frac{1}{4}}}{\sqrt{\pi}} e^{-(y-\frac{1}{2})^2} \leq 1 \quad \text{for } c = \frac{8 e^{\frac{1}{4}}}{\sqrt{\pi}} \approx 5,80 \end{aligned}$$

1a fortsat

$$Y = \sum_{k=1}^3 Z_k, \quad Z_k \sim e(1), \quad Z_k = -\ln V_k, \quad k=1, 2, 3$$

nægt.

$$Y = -\sum_{k=1}^3 \ln V_k \quad \left. \right\} \text{nægt.}$$

$$U \sim U[0, 1]$$

$$X = y, \text{ når } u \leq e^{-(y-\frac{1}{2})^2},$$

ellers opvis