8.2

4 c $P^T - I = \begin{bmatrix} -0,2 & 0,2 & 0 \\ 0,1 & -0,8 & 0,6 \\ 0,1 & 0,2 & -0,2 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & -3,4 \\ 0 & 1 & -1,2 \\ 0 & 0 & 0 \end{bmatrix}$

$\underline{\pi} = r(3,4;\ 1,2;\ 1)$, $\sum_i \pi_i = 1 \Rightarrow r = \frac{1}{5,8} = \frac{5}{29}$

$\underline{\pi} = \left(\frac{17}{29}, \frac{6}{29}, \frac{5}{29} \right)$

$E_H[\tau_H] = \frac{1}{\pi_H} = \frac{29}{5} = 5,8$

5 Fra eks. 8.4 side 448 – 449:

Markovkæde $\{X_n\}$, $S_X = \{0,1\}$, $P_X = [p_{ij}] = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$

Sæt $Y_n = (X_n, X_{n+1})$, $n = 0,1,\dots$, $S_Y = \{0,1\}^2$

$P(Y_{n+1} = (j,k) \mid Y_0 = (i_0,i_1), Y_1 = (i_1,i_2), \dots, Y_{n-1} = (i_{n-1}, i), Y_n = (i,j))$

$= P(X_{n+1} = j, X_{n+2} = k \mid X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}, X_n = i, X_{n+1} = j)$

$= P(X_{n+1} = j, X_{n+2} = k \mid X_n = i, X_{n+1} = j)$

$= P(Y_{n+1} = (j,k) \mid Y_n = (i,j))$

dvs. $\{Y_n\}$ opfylder Markovbetingelsen

Desuden ses, at $p_{(i,j),(j,k)} = p_{jk}$,

og $p_{(i,j),(m,k)} = 0$ for $m \neq j$

$P_Y = \begin{bmatrix} 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \end{bmatrix}$ (rækkefølge: $(0,0), (0,1), (1,0), (1,1)$)

$P_Y^T - I = \begin{bmatrix} -p & 0 & 1-p & 0 \\ p & -1 & p & 0 \\ 0 & q & -1 & q \\ 0 & 1-q & 0 & -q \end{bmatrix} \sim \dots \sim \begin{bmatrix} p & 0 & -(1-p) & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1-q & -q \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\underline{\pi} = r_1 \left(\frac{q}{p}(1-p), q, q, 1-q \right) = r\left(q(1-p), pq, pq, p(1-q) \right)$

$\sum_i \pi_i = 1 \Rightarrow r = \frac{1}{p+q} \Rightarrow \underline{\pi} = \left(\frac{q(1-p)}{p+q}, \frac{pq}{p+q}, \frac{pq}{p+q}, \frac{p(1-q)}{p+q} \right)$

8.2

8 Irreducibel Markovkæde, S endelig, $\forall i,j \in S : r_{ij} = r_{ji}$,

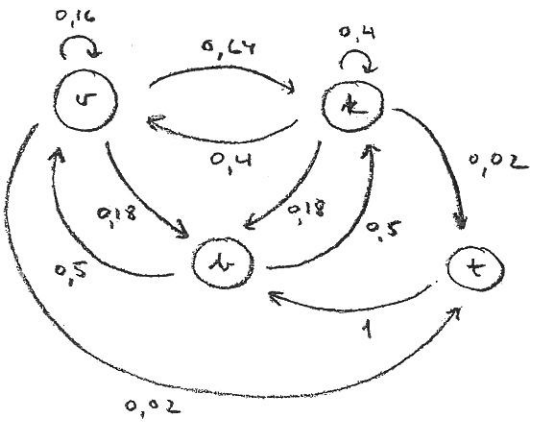
P kaldes dobbelt stokastisk, da $\sum_{j=1}^{m} r_{ij} = 1$ og $\sum_{i=1}^{m} r_{ij} = \sum_{i=1}^{m} r_{ji} = 1$

$(P^T - I)\,\underline{\pi}^T = \underline{0} \Leftrightarrow (P - I)\,\underline{\pi}^T = \underline{0}$ har løsning $\underline{\pi} = r\,\underline{1}_m$,

da $(P - I)\,\underline{1}_m = \left(\sum_{j=1}^{m} r_{ij} - 1, \ldots, \sum_{j=1}^{m} r_{mj} - 1\right) = \underline{0}$

$\sum_i \pi_i = 1 \Rightarrow r = \frac{1}{m} \Rightarrow \underline{\pi} = \left(\frac{1}{m}, \ldots \frac{1}{m}\right)$

11



fortsættes

8.2

11 fortsat

a $$P = \begin{bmatrix} 0,16 & 0,64 & 0,18 & 0,02 \\ 0,4 & 0,4 & 0,18 & 0,02 \\ 0,5 & 0,5 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P^T - I = \begin{bmatrix} -0,84 & 0,4 & 0,5 & 0 \\ 0,64 & -0,6 & 0,5 & 0 \\ 0,18 & 0,18 & -1 & 1 \\ 0,02 & 0,02 & 0 & -1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1,48 & 1 & 0 & 0 \\ -0,496 & 0 & 1 & 0 \\ -0,0496 & 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\pi} = r\,(1\,;\ 1,48\,;\ 0,496\,;\ 0,0496)$$

$$\sum_i \pi_i = 1 \Rightarrow r = \frac{1}{3,0256}$$

$$\underline{\pi} = (0,3305\,;\ 0,4892\,;\ 0,1639\,;\ 0,01639)$$

b $$E_t[N_b] = \frac{\pi_b}{\pi_t} = \frac{0,496}{0,0496} = 10$$

c Der er ikke taget højde for inddeling i ekspert.

12 Betjeningssted (én kunde ad gangen) med plads til tre i en kø.

Begivenhed: Afslutning af betjening eller ankomst af kunde(r).

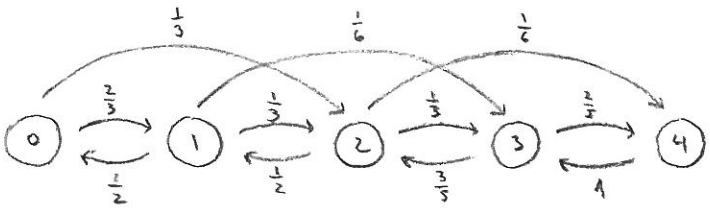
Ankomster: Sandsynlighed $\frac{2}{3}$ for alenekunde.

Sandsynlighed $\frac{1}{3}$ for parvis ankomst. Hvis ikke der er plads til begge, går begge (tæller i så fald ikke med som begivenhed).

X_n: Antal kunder i kø eller under betjening umiddelbart efter en begivenhed.

$S = \{0, 1, 2, 3, 4\}$

a



$$\frac{r_{34}}{r_{32}} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3} = \frac{\frac{2}{5}}{\frac{3}{5}}$$

fortsættes

8.2

12 a forts.

$$P = \begin{bmatrix} 0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{6} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{6} \\ 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$P^T - I = \begin{bmatrix} -1 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{2}{3} & -1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & -1 & \frac{3}{5} & 0 \\ 0 & \frac{1}{6} & \frac{1}{3} & -1 & 1 \\ 0 & 0 & \frac{1}{6} & \frac{2}{5} & -1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -6 & 6 \\ 0 & 0 & 1 & -48 & 84 \\ 0 & 0 & 0 & 14 & -25 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{\pi} = r\,(9, 18, 24, 25, 14), \quad \sum_i \pi_i = 1 \Rightarrow r = \frac{1}{90}$$

$$\underline{\pi} = \left(\frac{1}{10}, \frac{1}{5}, \frac{4}{15}, \frac{5}{18}, \frac{7}{45}\right)$$

b $P(\text{ing. kunder}) = \pi_0 = \frac{1}{10} = 0{,}1$

$P(\text{hoem fuld}) = \pi_4 = \frac{7}{45} = 0{,}1556$

c $$E_0[N_4] = \frac{\frac{7}{45}}{\frac{1}{10}} = \frac{70}{45} = \frac{14}{9} = 1{,}556$$

8.2

13 Grænsefordeling $\underline{q} = (q_1, \dots, q_r)$ for $\{X_n\}$

$$\forall j \in S: \quad P(X_{n+1} = j) = \sum_{i=1}^{r} P(X_{n+1} = j \mid X_n = i)\, P(X_n = i)$$

$$= \sum_{i=1}^{r} p_{ij}\, P(X_n = i)$$

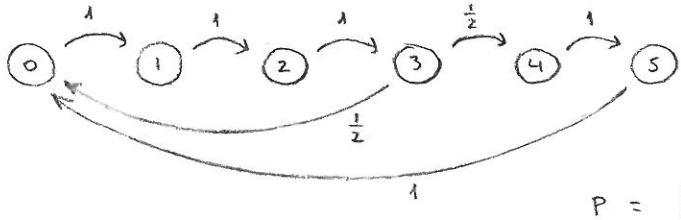
$$\Rightarrow \lim_{n\to\infty} P(X_{n+1} = j) = \lim_{n\to\infty} \sum_{i=1}^{r} p_{ij}\, P(X_n = i)$$

$$= \sum_{i=1}^{r} p_{ij} \lim_{n\to\infty} P(X_n = i)$$

$$\Leftarrow \quad q_j = \sum_{i=1}^{r} q_i p_{ij}$$

$\Leftrightarrow \underline{q} = P\underline{q} \Leftrightarrow \underline{q}$ er stationær fordeling

14



$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a 4

b sfd $\{4, 6\} = 2$

c $$P^T - I = \begin{bmatrix} -1 & 0 & 0 & \frac{1}{2} & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\underline{\pi} = r(2, 2, 2, 2, 1, 1)$, $\sum_i \pi_i = 1 \Rightarrow r = \frac{1}{10}$

$\underline{\pi} = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{10}, \frac{1}{10}\right)$, ikke grænsefordeling

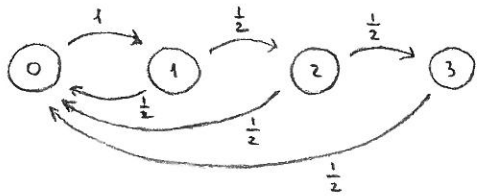
15 $\pi_k = \frac{1}{2^{k+1}}$, $k = 0, 1, \dots$, jf. eks. 8.16 side 463

a $E_{10}[\tau_{10}] = \frac{1}{\pi_{10}} = 2^{11} = 2048$

b $E_{10}[N_9] = \frac{\pi_9}{\pi_{10}} = \frac{2^{11}}{2^{10}} = 2$

8.2

16



$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & \cdots \\ \frac{1}{2} & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Kaden er irreducibel

$$P^T - I = \begin{bmatrix} -1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \cdots \\ 1 & -1 & 0 & 0 & \cdots \\ 0 & \frac{1}{2} & -1 & 0 & \cdots \\ 0 & 0 & \frac{1}{2} & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -\frac{1}{2} & 1 & 0 & \cdots \\ 0 & 0 & -\frac{1}{2} & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\pi_1 = \pi_0, \quad \pi_2 = \frac{1}{2}\pi_1 = \frac{1}{2}\pi_0, \quad \pi_k = \frac{1}{2}\pi_{k-1} = \frac{1}{2^{k-1}}\pi_0, \quad k = 1,2,\ldots$$

$$\sum_{k=0}^{\infty} \pi_k = 1 \Rightarrow \pi_0 + \pi_0 \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 1 \Rightarrow \pi_0 \left(1 + \sum_{k=0}^{\infty} \frac{1}{2^k}\right) = 1$$

$$\Rightarrow \pi_0 (1+2) = 1 \Rightarrow \pi_0 = \frac{1}{3}$$

$$\pi_0 = \frac{1}{3}, \quad \pi_k = \frac{1}{3}\frac{1}{2^{k-1}}, \quad k = 1,2,\ldots$$

Kaden er pos. rekurrent

$$d(0) = \text{sfd}\{2,3,\ldots\} = 1$$

Kaden er aperiodisk

Sammenfattende: Kaden er ergodisk
Grænsefordelingen eksisterer, $\underline{q} = \underline{\pi}$

17 $\{X_n\}$ ergodisk Markovkæde, n 'stor'

Betragt sekvensen af besøgte tilstande i omvendt tid, dvs. en kæde med overgangssandsynligheder $q_{ij} = P(X_{n-1} = j \mid X_n = i)$.

a
$$q_{ij} = P(X_{n-1} = j \mid X_n = i)$$
$$= \frac{P(X_{n-1} = j, X_n = i)}{P(X_n = i)}$$
$$= \frac{P(X_n = i \mid X_{n-1} = j)\, P(X_{n-1} = j)}{P(X_n = i)}$$
$$= \frac{\pi_j p_{ji}}{\pi_i}$$

fortsættes

8.2

17 a fortsat

At kæden i omvendt tid opfylder Markov-betingelsen, ses af

$$P(X_{n-1}=j \mid X_n=i,\ X_{n+1}=i_1,\ \dots,\ X_{n+k}=i_k)$$

$$= \frac{P(X_{n-1}=j,\ X_n=i,\ X_{n+1}=i_1,\ \dots,\ X_{n+k}=i_k)}{P(X_n=i,\ X_{n+1}=i_1,\ \dots,\ X_{n+k}=i_k)}$$

$$= \frac{P(X_{n+1}=i_1,\dots,X_{n+k}=i_k \mid X_n=i, X_{n-1}=j)\ P(X_n=i, X_{n-1}=j)}{P(X_{n+1}=i_1,\dots,X_{n+k}=i_k \mid X_n=i)\ P(X_n=i)}$$

$$= \frac{P(X_n=i,\ X_{n-1}=j)}{P(X_n=i)} = P(X_{n-1}=j \mid X_n=i)$$

b Kæden reversibel, når $p_{ij}=q_{ij}$ for alle i,j

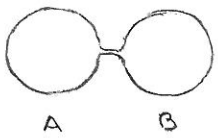
$$p_{ij}=q_{ij} \iff p_{ij}=\frac{\pi_j p_{ji}}{\pi_i} \iff \pi_i p_{ij}=\pi_j p_{ji}$$

c $\pi_i p_{ij}=\pi_j p_{ji} \Rightarrow \sum_i \pi_i p_{ij} = \sum_i \pi_j p_{ji}$

$$\iff (\pi P)_j = \pi_j 1 \iff \underline{\pi} P = \underline{\pi}$$

dvs. $\underline{\pi}$ er stationær fordeling

20

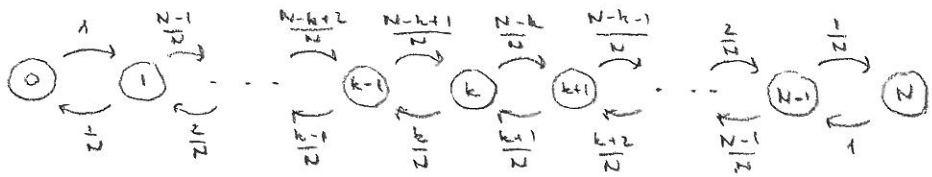


Totalt antal molekyler i to beholdere: N

I hver tidsenhed bevæger der sig tilfældigt ét molekyle fra den ene beholder til den anden.

X_n: Antal molekyler i beholder A

a.



b Hvert molekyle har sands. $\frac{1}{2}$ for at være i beholder A, dvs $\pi_k = \binom{N}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{N-k} = \binom{N}{k}\left(\frac{1}{2}\right)^N$

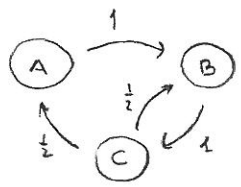
$$\pi_k \pi_{k,k+1} = \pi_{k+1}\pi_{k+1,k} \iff \binom{N}{k}\left(\frac{1}{2}\right)^N \frac{N-k}{N} = \binom{N}{k+1}\left(\frac{1}{2}\right)^N \frac{k+1}{N}$$

$$\iff \binom{N}{k} = \frac{k+1}{N-k}\binom{N}{k+1} \iff \binom{N}{k} = \binom{N}{k}$$

c $\underline{\pi}$ er ikke grænsefordeling, da kæden er periodisk med perioden 2

SLB

5.7



$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

i $R_{AA} = \{3, 5, \dots\}$ $R_{BA} = \{2, 4, 5, \dots\}$ $R_{CA} = \{1, 3, 4, 5, \dots\}$

$R_{AB} = \{1, 3, 4, 5, \dots\}$ $R_{BB} = \{2, 3, 5, \dots\}$ $R_{CB} = \{1, 2, 3, 4, 5, \dots\}$

$R_{AC} = \{2, 4, 5, \dots\}$ $R_{BC} = \{1, 3, 4, 5, \dots\}$ $R_{CC} = \{2, 3, 4, 5, \dots\}$

$\forall (i,j) : r_{ij}^{(s)} > 0$

ii $P^2 = \begin{bmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ $P^4 = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$

$P^5 = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{2} & \frac{3}{8} \\ \frac{1}{8} & \frac{3}{8} & \frac{1}{2} \end{bmatrix}$ $\forall (i,j) : r_{ij}^{(5)} > 0$

8.4

$$P^T - I = \begin{bmatrix} -1 & 0 & \frac{1}{2} \\ 1 & -1 & \frac{1}{2} \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\underline{\pi} = r(1, 2, 2)$, $\sum_i \pi_i = 1 \Rightarrow r = \frac{1}{5}$

$\underline{\pi} = \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$

$\underline{w} = \left(5, \frac{5}{2}, \frac{5}{2}\right)$