

8.2

21 $\{X_n\}$ irreducible, pos. recurrent m. stat. Jord. II

Med $g: S \rightarrow \mathbb{R}$ gælder $\frac{1}{n} \sum_{k=1}^n g(X_k) \xrightarrow{P} \sum_{j \in S} g(j) \pi_j$
 uanset valg af startford.

$$g(i) = P(X_k=i) \Rightarrow \frac{1}{n} \sum_{k=1}^n P(X_k=i) \xrightarrow{P} \sum_{j \in S} S_{ij} \pi_j = \pi_i$$

8.3

22 Simpelt random walk (for sym. r.w. sat $p=\frac{1}{2}$)

$$P(X_k=1) = p, \quad P(X_k=-1) = 1-p, \quad X_n \text{ er en waf.}$$

$$E X_k = 1p + (-1)(1-p) = 2p - 1$$

$$\text{Var } X_k = E X_k^2 - (E X_k)^2 = 1^2 p + (-1)^2 (1-p) - (2p-1)^2 = 1 - (2p-1)^2$$

$$S_m = \sum_{k=1}^m X_k$$

$$S_{m+n} = \sum_{k=1}^{m+n} X_k$$

$$E S_m = m(2p-1)$$

$$E S_{m+n} = (m+n)(2p-1)$$

$$\text{Var } S_m = m(1 - (2p-1)^2)$$

$$\text{Var } S_{m+n} = (m+n)(1 - (2p-1)^2)$$

$$\begin{aligned} \text{Cov}[S_m, S_{m+n}] &= \text{Cov}[S_m, S_m + \sum_{k=n+1}^{m+n} X_k] \\ &= \text{Var } S_m + \text{Cov}[\sum_{k=1}^m X_k, \sum_{k=n+1}^{m+n} X_k] \\ &= \text{Var } S_m + 0 = m(1 - (2p-1)^2) \end{aligned}$$

$$\begin{aligned} \rho(S_m, S_{m+n}) &= \frac{m(1 - (2p-1)^2)}{\sqrt{m(1 - (2p-1)^2)} \sqrt{(m+n)(1 - (2p-1)^2)}} \\ &= \frac{m}{\sqrt{m(m+n)}} = \frac{1}{\sqrt{1 + \frac{m}{n}}} \end{aligned}$$

$$\rho \rightarrow 0 \text{ for } \begin{cases} m \rightarrow \infty \\ n \text{ fast} \end{cases}$$

$$\rho \rightarrow 1 \text{ for } \begin{cases} m \rightarrow \infty \\ n \text{ lang} \end{cases}$$

23 Simpelt random walk, $p \neq \frac{1}{2}$

$$E S_m = E \left[\sum_{k=1}^m X_k \right] = m(1p + (-1)(1-p)) = m(2p-1)$$

$$\frac{1}{n} S_n \xrightarrow{P} 2p-1 \Rightarrow S_n \xrightarrow{P} \begin{cases} -\infty \text{ for } p < \frac{1}{2} \\ \infty \text{ for } p > \frac{1}{2} \end{cases}$$

Simpelt symmetrisk random walk med reflekterende barrierer i 0 og a

a



Irreductibel, pos. rekurrent, aperiodisk ($\pi_{00} > 0$),
dvs. ergodisk

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

P er symmetrisk,
dvs. den stationær
fordeling π kan ned-
skrives direkte, jf.
ørg. 8.2 8.

[Alternativt:

$$P^T - I = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

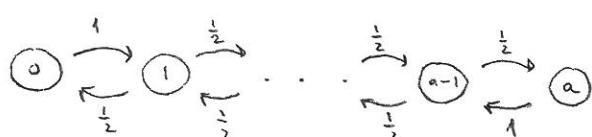
$$\pi = \pi_0 (1, 1, \dots, 1), \sum_i \pi_i = 1 \Rightarrow \pi_0 = \frac{1}{a+1}$$

$$\pi = \frac{1}{a+1} (1, 1, \dots, 1), \text{ også grænsefjord.}$$

b

$$E_0(\pi_0) = a+1$$

c



Irreductibel, pos. rekurrent,
periodisk med perioden 2

fortsættes

24 c forsat

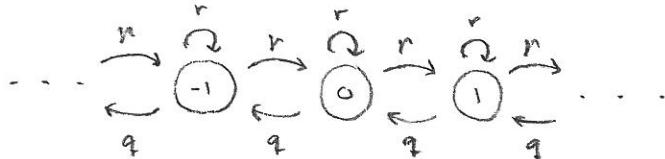
$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P^T - I = \begin{bmatrix} -1 & \frac{1}{2} & 0 & \dots & 0 & 0 & 0 \\ 1 & -1 & \frac{1}{2} & \dots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \dots & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \dots & \frac{1}{2} & -1 & 1 \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{2} & -1 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

$$\pi = \pi_0 (1, 2, \dots, 2, 1), \quad \sum_i \pi_i = 1 \Rightarrow \pi_0 = \frac{1}{2+2(a-1)}$$

$$\pi = \frac{1}{2a} (1, 2, \dots, 2, 1), \quad \text{ihre grauendorf.}$$

25



$$\text{i) Sat } u = P_0(\tau, < \infty)$$

$$\begin{aligned} \text{Betingning girer } u &= 1_p + r u + q u^2, \quad r = 1 - (n+q) \\ &\Rightarrow u^2 - (\frac{r}{q} + 1) u + \frac{1}{q} = 0 \Rightarrow u = \begin{cases} \frac{r}{q} \\ 1 \end{cases} \end{aligned}$$

$$P_0(\tau, < \infty) = \begin{cases} \frac{r}{q} & \text{for } r < q \\ 1 & \text{for } r \geq q \end{cases}$$

$$\text{ii) Sat } \mu = E_0[\tau]$$

$$\begin{aligned} \text{Betingning girer } \mu &= 1_p + (1+r)r + (1+2r)q \\ &\Rightarrow (p-q)\mu = 1 \end{aligned}$$

$$E_0[\tau] = \begin{cases} \frac{1}{r-q} & \text{for } r > q \\ \infty & \text{for } r \leq q \end{cases}$$

- 26 Simpelt random walk m. $\mu > \frac{1}{2}$, start i tilst. 0
 τ_r : tid til første besøg i tilst. $r \geq 1$

$$E_0[\tau_r] = r E_0[\tau_1] = r\mu$$

$$\text{Betingning: } \mu = 1_p + (1+2\mu)q \Rightarrow \mu = \frac{1}{p-q}$$

$$E_0[\tau_r] = \frac{r}{p-q}$$

- 27 Simpelt random walk m. $\mu \neq \frac{1}{2}$, start i 0
 $\tau_0 = \min \{ n \geq 1 : S_n = 0 \}$, tid til første besøg i 0

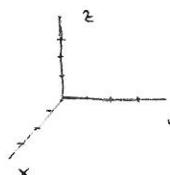
$$P_0(\tau_0 < \infty) = P_+(\tau_0 < \infty) \mu + P_-(\tau_0 < \infty) (1-\mu)$$

$$= \begin{cases} \frac{1-p}{1-(1-p)} n + 1(1-n) & \text{for } p > \frac{1}{2} \\ 1_p + \frac{p}{1-p}(1-p) & \text{for } p < \frac{1}{2} \end{cases}$$

$$= \begin{cases} 2(1-p) & \text{for } p > \frac{1}{2} \\ 2p & \text{for } p < \frac{1}{2} \end{cases}$$

$$= 2 \min \{ 1-p, p \}$$

29



Aantal skridt langs de tre aksler: n

$$\underline{S}_n = (S_i^{(x)}, S_j^{(y)}, S_k^{(z)}) , \quad i+j+k=n$$

$$\underline{S}_n \sim n \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right)$$

Ved start i 0 må der foretages lige mange skridt i pos. retning og i neg. retning på hver af akserne for at komme tilbage til 0

$$P_0(\underline{S}_{2n} = \underline{0}) = 0$$

$$P_0(\underline{S}_{2n} = \underline{0}) = \sum_{i+j+k=n} \binom{2n}{i,j,k,i,j,k} \left(\frac{1}{2}\right)^{2n}$$

$$= \sum_{i+j+k=n} \frac{(2n)!}{(i!)^2 (j!)^2 (k!)^2} \left(\frac{1}{2}\right)^{2n} = \sum_{i+j+k=n} \frac{(2n)!}{12^n i! j! k! n!} \binom{n}{i,j,k} \left(\frac{1}{2}\right)^n$$

$$\leq \frac{(2n)!}{12^n \left[\left(\frac{n}{3}\right)!\right]^3 n!} \approx \frac{(2n)^{2n} \sqrt{2n} e^{-2n} \sqrt{2\pi}}{12^n \left[\left(\frac{n}{3}\right)^{\frac{n}{3}} \sqrt{\frac{n}{3}} e^{-\frac{n}{3}} \sqrt{2\pi}\right]^3 n^n \sqrt{n} e^{-n} \sqrt{2\pi}}$$

$$= \frac{1}{2} \left(\frac{3}{\pi}\right)^{\frac{3}{2}} n^{-\frac{3}{2}} < \infty \text{ for alle } n \Rightarrow \underline{0} \text{ er transient}$$

$$\Rightarrow \{S_n\} \text{ er transient}$$

8.3

31 $P(X=k) = \frac{1}{3} \left(\frac{2}{3}\right)^k, k=0,1,\dots, EX = \mu = 3-1 = 2$

a $E[Z_n] = \mu^n = 2^n$

b $G(s) = \sum_{k=0}^{\infty} s^k \frac{1}{3} \left(\frac{2}{3}\right)^k = \frac{1}{3} \sum_{k=0}^{\infty} \left(\frac{2s}{3}\right)^k = \frac{1}{3} \frac{1}{1-\frac{2s}{3}} = \frac{1}{3-2s}$

$$G_2(s) = (G \circ G)(s) = \frac{1}{3-2s} \frac{1}{3-2s} = \frac{3-2s}{7-6s}$$

$$P(Z_2=0) = G_2(0) = \frac{3}{7}$$

c $G(s) = s \Leftrightarrow \frac{1}{3-2s} \Rightarrow 2s^2 - 3s + 1 = 0 \Rightarrow s = \begin{cases} 1 \\ \frac{1}{2} \end{cases}$

32 Ablauf X m. Ford. p_0, p_1, \dots , sandst. fr. br. fkt. $G(s)$, $EX = \mu$

Immigration Y m. Ford. q_0, q_1, \dots , sandst. fr. br. fkt. $H(s)$, $EY = \nu$

$$Z_0 = Y_0, Z_0 + 0; Z_1 = Y_1 + \sum_{k=1}^{Z_0} X_k; \dots; Z_n = Y_n + \sum_{k=1}^{Z_{n-1}} X_k$$

a $G_{Z_n}(s) = H(s) (H \circ G)(s)$

$$P(Z_1=0) = G_{Z_1}(0) = H(0) (H \circ G)(0) = q_0 H(p_0)$$

b $Z_n = Z_n^{(0)} + Z_n^{(1)} + \dots + Z_n^{(n)}$

$$G_{Z_n}(s) = \prod_{j=0}^n (H \circ G_{n-j})(s) = \prod_{j=0}^n (H \circ G_j)(s), G_0(s) = s$$

$$\begin{aligned} E[Z_n] &= G'_{Z_n}(1) = \sum_{k=0}^n (H \circ G_k)'(1) G'_k(1) \prod_{j \neq k} (H \circ G_j)'(1) \\ &= \sum_{k=0}^n \nu p^k = \nu \frac{1-p^{n+1}}{1-p} \end{aligned}$$

c $X \sim \text{Exp}(1, \mu), G(s) = 1 - \mu - \mu s, G^{\circ i}(s) = 1 - \mu^i + \mu^i s$

$$Y \sim \text{Exp}(\lambda), H(s) = e^{\lambda(s-1)}$$

$$G_{Z_n}(s) = \prod_{j=0}^n (H \circ G^{\circ j})(s) = \prod_{j=0}^n e^{\lambda(1-\mu^j + \mu^j s)}$$

$$= \prod_{j=0}^n e^{\lambda \mu^j (s-1)} = e^{\lambda \frac{1-\mu^{n+1}}{1-\mu} (s-1)}$$

$$Z_n \sim \text{Exp} \left(\lambda \frac{1-\mu^{n+1}}{1-\mu} \right)$$

9.3

$$\text{Markovkedede : } \pi_{i,0} = \frac{1}{i+2}, \quad \pi_{i,i+1} = \frac{i+1}{i+2}$$

$$\text{i} \quad f_{00}^{(n)} = \frac{1}{2} \cdot \frac{2}{3} \cdot \dots \cdot \frac{n+1}{n+2} = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}, \quad n=1, 2, \dots$$

$$\begin{aligned} \text{ii} \quad f_{00} &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \dots \\ &= \lim_{N \rightarrow \infty} \left(1 - \frac{1}{N+1} \right) = 1 \end{aligned}$$

$$\begin{aligned} \text{iii} \quad m_{00} &= \sum_{N=0}^{\infty} P(T_0 > N) = \sum_{N=0}^{\infty} (1 - P(T_0 \leq N)) \\ &= \sum_{N=0}^{\infty} \left(1 - \left(1 - \frac{1}{N+1} \right) \right) = \sum_{N=0}^{\infty} \frac{1}{N+1} = \infty \end{aligned}$$

iv Nulrekurrente

v Ingen stationær fordeling

9.4



$$\text{i} \quad f_{00} = 1$$

$$\text{ii} \quad m_{00} = \sum_{j=0}^{\infty} (1+j) u_j = \sum_{j=0}^{\infty} u_j + \sum_{j=0}^{\infty} j u_j = 1 + \mu$$

$$\text{iii} \quad P = \begin{bmatrix} u_0 & u_1 & u_2 & u_3 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$$P^T - I = \begin{bmatrix} u_0 - 1 & 1 & 0 & 0 & \dots \\ u_1 & -1 & 1 & 0 & \dots \\ u_2 & 0 & -1 & 1 & \dots \\ u_3 & 0 & 0 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ \sum u_j & -1 & 0 & 0 & \dots \\ \sum u_j & 0 & -1 & 0 & \dots \\ \sum u_j & 0 & 0 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

$$\underline{\pi} = \pi_0 (1, \sum u_i, \sum u_j, \dots)$$

$$\sum_i \pi_i = 1 \Rightarrow \pi_0 (1 + \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} u_j) = \pi_0 (1 + \sum_{j=1}^{\infty} j u_j)$$

$$= \pi_0 (1 + \mu) = 1 \Rightarrow \pi_0 = \frac{1}{1 + \mu}, \quad \mu < \infty$$

$$\underline{\pi} = \frac{1}{1 + \mu} (1, \sum_{j=1}^{\infty} u_j, \sum_{j=2}^{\infty} u_j, \dots, \sum_{j=n}^{\infty} u_j, \dots), \quad \mu < \infty$$