Integraludregninger

Ligefordelingen, $X \sim U[a;b]$

Tæthedsfunktion:

$$f(x) = \frac{1}{b-a}, \quad a \le x \le b$$

Kontrol:

$$\int_a^b \frac{1}{b-a}\,\mathrm{d}x = \frac{1}{b-a}\big[x\big]_a^b = \frac{1}{b-a}(b-a) = 1$$

Middelværdi og varians:

$$\mathrm{E}X = \int_a^b x\,\frac{1}{b-a}\,\mathrm{d}x = \frac{1}{b-a}\Big[\frac{x^2}{2}\Big]_a^b = \frac{1}{b-a}\,\frac{b^2-a^2}{2} = \frac{a+b}{2}$$

$$\mathrm{E}[X^2] = \int_a^b x^2\frac{1}{b-a}\,\mathrm{d}x = \frac{1}{b-a}\Big[\frac{x^3}{3}\Big]_a^b = \frac{1}{b-a}\,\frac{b^3-a^3}{3} = \frac{b^2+ab+a^2}{3}$$

$$\mathrm{Var}X = \frac{b^2+ab+a^2}{3} - \Big(\frac{a+b}{2}\Big)^2 = \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

Eksponentialfordelingen, $X \sim e(\lambda)$

Tæthedsfunktion:

$$f(x) = \lambda\,e^{-\lambda x}, \quad 0 \le x < \infty$$

Kontrol:

$$\int_0^\infty \lambda\,e^{-\lambda x}\,\mathrm{d}x = \lambda\Big[-\frac{1}{\lambda}e^{-\lambda x}\Big]_0^\infty = 1$$

Middelværdi og varians:

$$\mathrm{E}X = \int_0^\infty P(X > x)\,\mathrm{d}x\;^{1)} = \int_0^\infty e^{-\lambda x}\,\mathrm{d}x = \Big[-\frac{1}{\lambda}e^{-\lambda x}\Big]_0^\infty = \frac{1}{\lambda}$$

$$\mathrm{E}[X^2] = \int_0^\infty 2\,x\,P(X > x)\,\mathrm{d}x = \frac{2}{\lambda}\int_0^\infty x\,\lambda\,e^{-\lambda x}\,\mathrm{d}x = \frac{2}{\lambda}\,\mathrm{E}X = \frac{2}{\lambda^2}$$

$$\mathrm{Var}X = \frac{2}{\lambda^2} - \Big(\frac{1}{\lambda}\Big)^2 = \frac{1}{\lambda^2}$$

1) Alternativt: $\mathrm{E}X = \int_0^\infty x\,\lambda\,e^{-\lambda x}\,\mathrm{d}x = \cdots = \frac{1}{\lambda}$, analogt $\mathrm{E}[X^2]$.

Normalfordelingen, $X \sim N(\mu, \sigma^2)$

Tæthedsfunktion:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big), \quad -\infty < x < \infty$$

Kontrol:

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sigma}\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)\,\mathrm{d}x, \qquad u=\tfrac{x-\mu}{\sigma},\ x=\sigma u+\mu,\quad \mathrm{d}u=\tfrac{1}{\sigma}\,\mathrm{d}x\\
&=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{u^2}{2}\Big)\,\mathrm{d}u=\Bigg(\bigg(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{u^2}{2}\Big)\,\mathrm{d}u\bigg)^2\Bigg)^{\frac{1}{2}}\\
&=\bigg(\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{u^2}{2}\Big)\,\mathrm{d}u\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{v^2}{2}\Big)\,\mathrm{d}v\bigg)^{\frac{1}{2}}\\
&=\bigg(\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\Big(-\frac{u^2+v^2}{2}\Big)\,\mathrm{d}u\,\mathrm{d}v\bigg)^{\frac{1}{2}}\\
&=\bigg(\frac{1}{2\pi}\int_{0}^{2\pi}\int_{0}^{\infty}\exp\Big(-\frac{r^2}{2}\Big)r\,\mathrm{d}r\,\mathrm{d}\theta\bigg)^{\frac{1}{2}}\\
&=\bigg(\int_{0}^{\infty}\exp\Big(-\frac{r^2}{2}\Big)\,\mathrm{d}\Big(\frac{r^2}{2}\Big)\bigg)^{\frac{1}{2}}=\bigg(\Big[-\exp\Big(-\frac{r^2}{2}\Big)\Big]_0^{\infty}\bigg)^{\frac{1}{2}}\\
&=1^{\frac{1}{2}}=1
\end{aligned}$$

Middelværdi og varians:

$$\begin{aligned}
\mathrm{E}X&=\int_{-\infty}^{\infty}x\frac{1}{\sqrt{2\pi}\sigma}\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)\,\mathrm{d}x, \qquad u=\tfrac{x-\mu}{\sigma},\ x=\sigma u+\mu,\quad \mathrm{d}u=\tfrac{1}{\sigma}\,\mathrm{d}x\\
&=\int_{-\infty}^{\infty}(\sigma u+\mu)\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{u^2}{2}\Big)\,\mathrm{d}u\\
&=\sigma\int_{-\infty}^{\infty}u\,\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{u^2}{2}\Big)\,\mathrm{d}u+\mu\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{u^2}{2}\Big)\,\mathrm{d}u\ ^{2)}\\
&=\sigma\Big[-\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{u^2}{2}\Big)\Big]_{-\infty}^{\infty}+\mu\cdot 1=\sigma\cdot 0+\mu=\mu
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}X&=\int_{-\infty}^{\infty}(x-\mu)^2\frac{1}{\sqrt{2\pi}\sigma}\exp\Big(-\frac{(x-\mu)^2}{2\sigma^2}\Big)\,\mathrm{d}x, \qquad u=\tfrac{x-\mu}{\sigma},\ x=\sigma u+\mu,\quad \mathrm{d}u=\tfrac{1}{\sigma}\,\mathrm{d}x\\
&=\int_{-\infty}^{\infty}(\sigma u)^2\frac{1}{\sqrt{2\pi}\sigma}\exp\Big(-\frac{u^2}{2}\Big)\,\mathrm{d}u\\
&=\sigma^2\int_{-\infty}^{\infty}u\cdot u\,\frac{1}{\sqrt{2\pi}\sigma}\exp\Big(-\frac{u^2}{2}\Big)\,\mathrm{d}u\\
&=\sigma^2\bigg(\Big[u\Big(-\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{u^2}{2}\Big)\Big)\Big]_{-\infty}^{\infty}+\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{u^2}{2}\Big)\,\mathrm{d}u\bigg)\\
&=\sigma^2(0+1)=\sigma^2
\end{aligned}$$

2) $\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{u^2}{2}\Big)$ er tæthedsfunktion for $U \sim N(0,1)$.

Lognormalfordelingen, $X \sim LN(\alpha, \beta)$**,** hvor $\ln X \sim N(\mu, \sigma^2)$

α og β står for hhv. middelværdi og varians i lognormalfordelingen.

Middelværdi og varians:

Bemærk, at $u = \ln x \Leftrightarrow x = e^u$.

$$\begin{aligned}
\mathrm{E}X &= \int_{-\infty}^{\infty} e^u \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{(u-\mu)^2}{2\sigma^2}\Big)\,\mathrm{d}u \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{u^2 - (2\mu + 2\sigma^2)u + \mu^2}{2\sigma^2}\Big)\,\mathrm{d}u \\
&= \exp\Big(\frac{2\mu\sigma^2 + \sigma^4}{2\sigma^2}\Big) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{(u - (\mu + \sigma^2))^2}{2\sigma^2}\Big)\,\mathrm{d}u \\
&= \exp\Big(\mu + \frac{\sigma^2}{2}\Big) \cdot 1 = e^{\mu + \frac{\sigma^2}{2}}, \quad \text{altså } \alpha = e^{\mu + \frac{\sigma^2}{2}}
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}[X^2] &= \int_{-\infty}^{\infty} e^{2u} \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{(u-\mu)^2}{2\sigma^2}\Big)\,\mathrm{d}u \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{u^2 - 2(\mu + 2\sigma^2)u + \mu^2}{2\sigma^2}\Big)\,\mathrm{d}u \\
&= \exp\Big(\frac{4\mu\sigma^2 + 4\sigma^4}{2\sigma^2}\Big) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\Big(-\frac{(u - (\mu + 2\sigma^2))^2}{2\sigma^2}\Big)\,\mathrm{d}u \\
&= \exp\Big(2\mu + 2\sigma^2\Big) \cdot 1 = e^{2\mu + 2\sigma^2}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}X &= e^{2\mu + 2\sigma^2} - \Big(e^{\mu + \frac{\sigma^2}{2}}\Big)^2 = e^{2\mu + \sigma^2} e^{\sigma^2} - e^{2\mu + \sigma^2} \\
&= e^{2\mu + \sigma^2}(e^{\sigma^2} - 1), \quad \text{altså } \beta = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1)
\end{aligned}$$

Gammafunktionen, $\Gamma(x) = \displaystyle\int_0^{\infty} t^{x-1} e^{-t}\,\mathrm{d}t,\ 0 < x < \infty$

Funktionalligning:

$$\begin{aligned}
\Gamma(x) &= \big[t^{x-1}(-e^{-t})\big]_{t=0}^{\infty} + \int_0^{\infty} (x-1) t^{x-2} e^{-t}\,\mathrm{d}t = 0 + (x-1)\int_0^{\infty} t^{(x-1)-1} e^{-t}\,\mathrm{d}t \\
&= (x-1)\,\Gamma(x-1) \quad \Rightarrow \quad \Gamma(x+1) = x\Gamma(x), \quad 0 < x < \infty
\end{aligned}$$

Specielt gælder:

$$\Gamma(n) = (n-1)(n-2)\cdots 2 \cdot 1 = (n-1)!\,, \qquad n \in \mathbb{N}$$

$$\begin{aligned}
\Gamma\Big(\frac{1}{2}\Big) &= \int_0^{\infty} t^{-\frac{1}{2}} e^{-t}\,\mathrm{d}t, \qquad t = \tfrac{u^2}{2},\ \mathrm{d}t = u\,\mathrm{d}u \\
&= \int_0^{\infty} \frac{\sqrt{2}}{u} \exp\Big(-\frac{u^2}{2}\Big) u\,\mathrm{d}u = \sqrt{\pi}\,2\int_0^{\infty} \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{u^2}{2}\Big)\,\mathrm{d}u = \sqrt{\pi}
\end{aligned}$$

$$\begin{aligned}\Gamma\Big(n+\frac{1}{2}\Big) &= \Big(n+\frac{1}{2}-1\Big)\Gamma\Big(n+\frac{1}{2}-1\Big) = \Big(n-\frac{1}{2}\Big)\Gamma\Big(n-\frac{1}{2}\Big) = \cdots \\ &= \Big(n-\frac{1}{2}\Big)\Big(n-\frac{3}{2}\Big)\cdots\frac{3}{2}\,\frac{1}{2}\,\Gamma\Big(\frac{1}{2}\Big) = \frac{1}{2^n}(2n-1)(2n-3)\cdots 3\cdot 1\,\sqrt{\pi} \\ &= \frac{1}{2^n}\,\frac{(2n)!}{2^n\,n!}\,\sqrt{\pi} = \frac{(2n)!}{4^n\,n!}\,\sqrt{\pi}, \quad n\in\mathbb{N}\end{aligned}$$

Gammafordelingen, $X \sim \Gamma(\alpha,\lambda)$

Tæthedsfunktion:

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\lambda x}, \quad 0\le x<\infty$$

Kontrol:

$$\int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\lambda x}\,\mathrm{d}x = \frac{1}{\Gamma(\alpha)}\int_0^\infty (\lambda x)^{\alpha-1}e^{-(\lambda x)}\,\mathrm{d}(\lambda x) = \frac{1}{\Gamma(\alpha)}\,\Gamma(\alpha) = 1$$

Middelværdi og varians:

$$\mathrm{E}X = \int_0^\infty x\,\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\lambda x}\,\mathrm{d}x = \frac{\alpha}{\lambda}\int_0^\infty \frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^{(\alpha+1)-1}e^{-\lambda x}\,\mathrm{d}x = \frac{\alpha}{\lambda}\cdot 1 = \frac{\alpha}{\lambda}$$

$$\begin{aligned}\mathrm{E}[X^2] &= \int_0^\infty x^2\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1}e^{-\lambda x}\,\mathrm{d}x = \frac{\alpha(\alpha+1)}{\lambda^2}\int_0^\infty \frac{\lambda^{\alpha+2}}{\Gamma(\alpha+2)} x^{(\alpha+2)-1}e^{-\lambda x}\,\mathrm{d}x \\ &= \frac{\alpha^2+\alpha}{\lambda^2}\cdot 1 = \frac{\alpha^2+\alpha}{\lambda^2}\end{aligned}$$

$$\mathrm{Var}X = \frac{\alpha^2+\alpha}{\lambda^2} - \Big(\frac{\alpha}{\lambda}\Big)^2 = \frac{\alpha}{\lambda^2}$$

Tæthedsfunktion og fordelingsfunktion, når α er heltallig ($\alpha := n$):

$$f(x) = \frac{\lambda^n}{(n-1)!} x^{n-1}e^{-\lambda x}, \quad 0\le x<\infty$$

$$\begin{aligned}F(x) &= \int_0^x \frac{\lambda^n}{(n-1)!} t^{n-1}e^{-\lambda t}\,\mathrm{d}t \\ &= \Big[\frac{\lambda^n}{(n-1)!} t^{n-1}\big(-\frac{1}{\lambda}e^{-\lambda t}\big)\Big]_0^x + \int_0^x \frac{\lambda^n}{(n-1)!}(n-1)t^{n-2}\frac{1}{\lambda}e^{-\lambda t}\,\mathrm{d}t \\ &= -\frac{\lambda^{n-1}}{(n-1)!} x^{n-1}e^{-\lambda x} + \int_0^x \frac{\lambda^{n-1}}{(n-2)!} t^{n-2}e^{-\lambda t}\,\mathrm{d}t \\ &\vdots \\ &= -\Big(\sum_{k=1}^{n}\frac{\lambda^{k-1}}{(k-1)!}x^{k-1}\Big)e^{-\lambda x} + \int_0^x \lambda e^{-\lambda t}\,\mathrm{d}t = -\Big(\sum_{k=0}^{n-1}\frac{\lambda^k}{k!}x^k\Big)e^{-\lambda x} + 1 \\ &= 1 - e^{-\lambda x}\sum_{k=0}^{n-1}\frac{\lambda^k x^k}{k!}, \quad 0\le x<\infty\end{aligned}$$

Cauchyfordelingen eller t-fordelingen med 1 frihedsgrad

Tæthedsfunktion:

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$

Kontrol:

$$\int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)}\, \mathrm{d}x = \frac{1}{\pi}\big[\arctan x\big]_{-\infty}^{\infty} = \frac{1}{\pi}\Big(\frac{\pi}{2}+\frac{\pi}{2}\Big) = 1$$

Middelværdi og varians er ikke definerede, idet

$$\int_{-\infty}^{\infty} x\,\frac{1}{\pi(1+x^2)}\, \mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, \mathrm{d}(1+x^2) = \frac{1}{2\pi}\big[\ln(1+x^2)\big]_{-\infty}^{\infty}$$

er divergent.

Weibullfordelingen, $X \sim W(\alpha, \lambda)$

Tæthedsfunktion:

$$f(x) = \lambda\,\alpha\,x^{\alpha-1}e^{-\lambda x^{\alpha}}, \quad 0 \le x < \infty$$

Kontrol:

$$\begin{aligned} &\int_0^{\infty} \lambda\,\alpha\,x^{\alpha-1}e^{-\lambda x^{\alpha}}\, \mathrm{d}x, \qquad && u = x^{\alpha} \\ & && \mathrm{d}u = \alpha\,x^{\alpha-1}\,\mathrm{d}x \\ &= \int_0^{\infty} \lambda\, e^{-\lambda u}\, \mathrm{d}u = 1 \end{aligned}$$

Middelværdi og varians:

$$\begin{aligned} \mathrm{E}X &= \int_0^{\infty} x\,\lambda\,\alpha\,x^{\alpha-1}e^{-\lambda x^{\alpha}}\, \mathrm{d}x, \qquad && u = \lambda\,x^{\alpha},\ \ x = \big(\tfrac{u}{\lambda}\big)^{\frac{1}{\alpha}} \\ & && \mathrm{d}u = \lambda\,\alpha\,x^{\alpha-1}\,\mathrm{d}x \\ &= \int_0^{\infty} \Big(\frac{u}{\lambda}\Big)^{\frac{1}{\alpha}} e^{-u}\, \mathrm{d}u \\ &= \Big(\frac{1}{\lambda}\Big)^{\frac{1}{\alpha}} \int_0^{\infty} u^{(1+\frac{1}{\alpha})-1}e^{-u}\, \mathrm{d}u \\ &= \Big(\frac{1}{\lambda}\Big)^{\frac{1}{\alpha}}\, \Gamma\Big(1+\frac{1}{\alpha}\Big) \end{aligned}$$

$$\begin{aligned} \mathrm{E}[X^2] &= \int_0^{\infty} x^2\lambda\,\alpha\,x^{\alpha-1}e^{-\lambda x^{\alpha}}\, \mathrm{d}x, \qquad && u = \lambda\,x^{\alpha},\ \ x = \big(\tfrac{u}{\lambda}\big)^{\frac{1}{\alpha}} \\ & && \mathrm{d}u = \lambda\,\alpha\,x^{\alpha-1}\,\mathrm{d}x \\ &= \int_0^{\infty} \Big(\frac{u}{\lambda}\Big)^{\frac{2}{\alpha}} e^{-u}\, \mathrm{d}u \\ &= \Big(\frac{1}{\lambda}\Big)^{\frac{2}{\alpha}} \int_0^{\infty} u^{(1+\frac{2}{\alpha})-1}e^{-u}\, \mathrm{d}u \\ &= \Big(\frac{1}{\lambda}\Big)^{\frac{2}{\alpha}}\, \Gamma\Big(1+\frac{2}{\alpha}\Big) \end{aligned}$$

$$\begin{aligned}\operatorname{Var}X &= \left(\frac{1}{\lambda}\right)^{\frac{2}{\alpha}}\Gamma\Big(1+\frac{2}{\alpha}\Big) - \left(\Big(\frac{1}{\lambda}\Big)^{\frac{1}{\alpha}}\Gamma\Big(1+\frac{1}{\alpha}\Big)\right)^2 \\ &= \left(\frac{1}{\lambda}\right)^{\frac{2}{\alpha}}\left(\Gamma\Big(1+\frac{2}{\alpha}\Big) - \Big(\Gamma\Big(1+\frac{1}{\alpha}\Big)\Big)^2\right)\end{aligned}$$

28.2.2011/BR