

Kursusindhold

- vektorrum
- matrixregning
- lineære ligningsystemer
- lineære transformationer
- determinanter
- egenverdier
- orthogonalitet

Vektorer i \mathbb{R}^n

$$\bar{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$$

v_1, v_2, \dots, v_n kaldes vektorens komponenter

Operationer på vektorer i \mathbb{R}^n , $\bar{u}, \bar{v} \in \mathbb{R}^n$, $r \in \mathbb{R}$

$$\bar{u} + \bar{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$$

$$r\bar{v} = (rv_1, rv_2, \dots, rv_n)$$

$$\bar{u} \cdot \bar{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$$\|\bar{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

$$d(\bar{u}, \bar{v}) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \dots + (u_n - v_n)^2}$$

$$\cos \theta = \frac{\bar{u} \cdot \bar{v}}{\|\bar{u}\| \|\bar{v}\|}$$

$$\bar{u}_{\bar{v}} = \frac{\bar{u} \cdot \bar{v}}{\|\bar{v}\|^2} \bar{v}$$

Vektorrum V

$\bar{u}, \bar{v} \in V$

$r \in S$ (et skalarlegeme, oftest \mathbb{R})

$(V, +)$ skal være en kommutativ gruppe, dvs.

$$\bar{u} + \bar{v} = \bar{v} + \bar{u} \quad (\text{kommutativ lov opfyldt})$$

$$(\bar{u} + \bar{v}) + \bar{w} = \bar{u} + (\bar{v} + \bar{w}) \quad (\text{associative lov opfyldt})$$

$$\exists \bar{0} \forall \bar{v}: \bar{v} + \bar{0} = \bar{v} \quad (\text{der findes et neutralt element})$$

$$\forall \bar{v} \exists -\bar{v}: \bar{v} + (-\bar{v}) = \bar{0} \quad (\text{enhver vektor har et invers element})$$

Kvar til skalarmultiplikation:

$$r(\bar{u} + \bar{v}) = r\bar{u} + r\bar{v}$$

$$(r+s)\bar{v} = r\bar{v} + s\bar{v}$$

$$r(s\bar{v}) = (rs)\bar{v}$$

$$1\bar{v} = \bar{v}$$

Eksempler på vektorrum

\mathbb{R}^n (det n-dimensionale talrum)

P_n (mængden af polynomier af højst n'te grad)

Mængde V, som ikke er et vektorrum

Addition defineres som sædvanlig, dvs.

$(V, +)$ er en kommutativ gruppe.

Skalarmult. defineres $r\bar{v} = \bar{0}$ for alle $r \in R$ og alle $\bar{v} \in V$

$$r(\bar{u} + \bar{v}) = \bar{0} = \bar{u} + \bar{v} = r\bar{u} + r\bar{v} \quad \text{ok}$$

$$(r+s)\bar{v} = \bar{0} = \bar{v} + \bar{v} = r\bar{v} + s\bar{v} \quad \text{ok}$$

$$r(s\bar{v}) = r\bar{0} = \bar{0} = (rs)\bar{v} \quad \text{ok}$$

$$1\bar{v} = \bar{v} \neq \bar{0} \quad \text{for } \bar{v} \neq \bar{0} \quad \text{hværet } 1\bar{v} = \bar{v} \text{ ikke opfyldt}$$

V er ikke et vektorrum

Matricer

Bemerk sprogbogen : Én matrix
Flere matricer

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \quad a_{ij} \text{ kaldes } i,j\text{'te element}$$

a_{ij} noteres også (A)_{ij}

A's format : m × n (m rækker og n søjler)

$$A = B \Leftrightarrow a_{ij} = b_{ij} \text{ for alle } i, j$$

nodvendig betingelse : A og B har samme format

Notation med søjlevectorer :

$$A = [\bar{a}_1 \ \bar{a}_2 \ \dots \ \bar{a}_n], \text{ hvor } \bar{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}, \quad j=1, \dots, n$$

Tilsvarende kaldes $[a_{i1} \ a_{i2} \ \dots \ a_{in}], \quad i=1, \dots, m$, for rækkevectorer

Nulmatrix : $O = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$

Kvadratisk matrix : $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}_{n \times n}$

øvre triangulær

$$\begin{bmatrix} \Delta \\ 0 \end{bmatrix}$$

nedre triangulær

$$\begin{bmatrix} 0 \\ \Delta \end{bmatrix}$$

diagonalmatrix

$$D = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix} = [d_{11} \ d_{22} \ \dots \ d_{nn}]$$

enhetsmatrix

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = [1 \ 1 \ \dots \ 1]$$

Operationer på matricer

- addition : $A + B = C$, $c_{ij} = a_{ij} + b_{ij}$ for alle i, j

eks : $\begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 1 & -2 \\ 5 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 9 & -1 & 2 \end{bmatrix}$

- skalarmult. : $rA = C$, $c_{ij} = r a_{ij}$ for alle i, j

eks : $3 \begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -6 & 3 & 9 \\ 12 & 0 & -3 \end{bmatrix}$

- transponering : $A^T = C$, $c_{ij} = a_{ji}$ for alle i, j

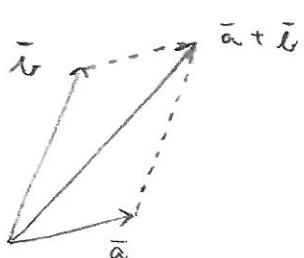
eks : $\begin{bmatrix} -2 & 1 & 3 \\ 4 & 0 & -1 \end{bmatrix}^T = \begin{bmatrix} -2 & 4 \\ 1 & 0 \\ 3 & -1 \end{bmatrix}$

bemerk $(A+B)^T = A^T + B^T$ | $(A^T)^T = A$

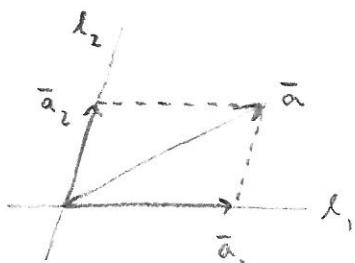
$(rA)^T = rA^T$ | $(AB)^T = B^T A^T$
(vises senere)

Vektorgeometri

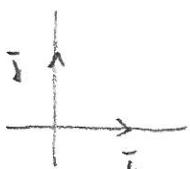
addition

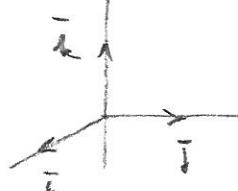


oplysning i komponenter



Naturlig (eller kanonisk) basis

- for \mathbb{R}^2 :  $\bar{i} = (1, 0)$
 $\bar{j} = (0, 1)$

- for \mathbb{R}^3 :  $\bar{i} = (1, 0, 0)$
 $\bar{j} = (0, 1, 0)$
 $\bar{k} = (0, 0, 1)$

- for \mathbb{R}^n : $\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n$; $\bar{e}_j = (0, \dots, 0, 1, 0, \dots, 0)$
 \uparrow
 $j^{\text{te}} \text{ komponent}$

bemerk: $\bar{a} = (a_1, a_2, \dots, a_n) = a_1 \bar{e}_1 + a_2 \bar{e}_2 + \dots + a_n \bar{e}_n$

Linearkombination af $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k \in \mathbb{R}^n$:

$$r_1 \bar{v}_1 + r_2 \bar{v}_2 + \dots + r_k \bar{v}_k, \text{ hvor } r_1, r_2, \dots, r_k \in \mathbb{R}$$

bemerk $r_1 \bar{v}_1 + r_2 \bar{v}_2 + \dots + r_k \bar{v}_k$

$$\underline{s_1 \bar{v}_1 + s_2 \bar{v}_2 + \dots + s_k \bar{v}_k}$$

$$(r_1 + s_1) \bar{v}_1 + (r_2 + s_2) \bar{v}_2 + \dots + (r_k + s_k) \bar{v}_k$$

dvs. en sum af linearkombinationer er selv en
linearkombination,
og dermed også en linearkombination af linearkombinationer er selv en linearkombination

Matrix - vektor - produkt, $A_{m \times n}, \bar{x} \in \mathbb{R}^n$

$$A\bar{x} = [\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = x_1 \bar{a}_1 + x_2 \bar{a}_2 + \dots + x_n \bar{a}_n,$$

dvs. den resulterende vektor er en linearkombination af A's sojlevektorer

konvention: når vektorer optræder i forbindelse med
matricer, opfatters de som sojlevektorer

$A\bar{x}$ skrevet helt ud:

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix} \quad \text{Søjlevektor} \quad (m \times 1)$$

(fremkommet ved række-søje-multiplikation)

eks.

$$\begin{bmatrix} 0,8 & 0,6 & 0,4 \\ 0,2 & 0,4 & 0,6 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0,8 \cdot 2 + 0,6 \cdot 5 + 0,4 \cdot 1 \\ 0,2 \cdot 2 + 0,4 \cdot 5 + 0,6 \cdot 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Regneregler for matrix-vektorprodukt

$$A(\bar{u} + \bar{v}) = A\bar{u} + A\bar{v}$$

$$A\bar{e}_j = \bar{e}_j$$

$$A(r\bar{v}) = r(A\bar{v}) = (rA)\bar{v}$$

$$A\bar{0} = \bar{0}$$

$$(A+B)\bar{v} = A\bar{v} + B\bar{v}$$

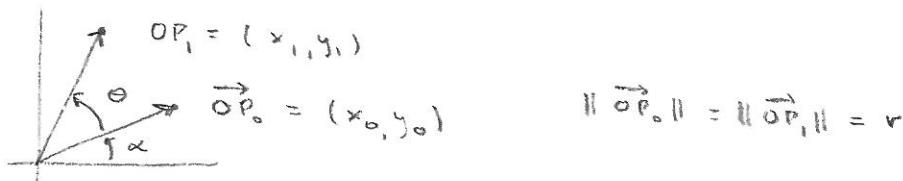
$$0\bar{v} = \bar{0}$$

$$A\bar{v} = B\bar{v} \text{ for alle } \bar{v} \Rightarrow A = B$$

$$1\bar{v} = \bar{v}$$

$$A(r_1\bar{v}_1 + r_2\bar{v}_2 + \dots + r_k\bar{v}_k) = r_1A\bar{v}_1 + r_2A\bar{v}_2 + \dots + r_kA\bar{v}_k$$

Rotation af vektor i \mathbb{R}^2 med vinklen θ



$$x_1 = r \cos(\alpha + \theta) = r (\cos \alpha \cos \theta - \sin \alpha \sin \theta)$$

$$= (r \cos \alpha) \cos \theta - (r \sin \alpha) \sin \theta = x_0 \cos \theta - y_0 \sin \theta$$

$$y_1 = r \sin(\alpha + \theta) = r (\sin \alpha \cos \theta + \cos \alpha \sin \theta)$$

$$= (r \sin \alpha) \cos \theta + (r \cos \alpha) \sin \theta = x_0 \sin \theta + y_0 \cos \theta$$

dvs.

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$\overset{\uparrow}{\bar{e}} \quad \overset{\uparrow}{\bar{f}}$

venstre, at $\hat{e} = \bar{f}$