

## Dimension af vektorrum - fortsat

$\dim V = \#$  basiselementer i basis for  $V$

Underrum hørende til en matris A, m x n

$$\text{Nul } A = \{ \bar{x} \mid A\bar{x} = \bar{0} \} \subseteq \mathbb{R}^n$$

$\Rightarrow$  basisvektor = nullität A ( $= n - \text{rang } A$ )

$$\dim \text{Nul } A = \text{nullität } A$$

$$\text{Col } A = \text{Sp} \{ \bar{a}_1, \dots, \bar{a}_n \} \subset \mathbb{R}^m$$

$\#$  Basisvektoren = rang A

$$\dim \text{Col } A = \text{rang } A$$

Row A = summen af værdierne i A  $\in \mathbb{R}^n$

Spændet af rækkevektoren ændres ikke ved rækkeoperationer, dvs.  $\text{Row A} = \text{Row R}$ , hvor  $R$  er den reducerede række-echelon-form.

# Basisvektoren = # pivotelementer i R = rang A

$$\dim \text{Row } A = \text{rank } A$$

$$\text{Col } A^T = \text{Row } A \quad \text{dis. dim Col } A^T = \text{rank } A$$

by derived  $\dim \text{Col } A = \dim \text{Col } A^T$

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$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\left[ \begin{array}{ccccc} -3 & 5 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 2 & -1 \end{array} \right] \xrightarrow{\text{Row Reduction}} \left[ \begin{array}{ccccc} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 3 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccccc} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

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Linn. meth.

$$\text{Løsn. til } A\bar{x} = \bar{0} : \quad \bar{x} = r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix},$$

$r, s, t \in \mathbb{R}$

$$\text{Nul } A = \text{Sp} \{(2, 1, 0, 0, 0), (1, 0, -2, 1, 0), (-3, 0, 2, 0, 1)\}$$

$$\mathcal{B}_{\text{Nul } A} = \{(2, 1, 0, 0, 0), (1, 0, -2, 1, 0), (-3, 0, 2, 0, 1)\}$$

$$\dim \text{Nul } A = 3 \quad (= \text{nullitet } A)$$

$$\text{Col } A = \text{Sp} \{(-3, -1, 2), (-1, 2, 5)\}$$

$$\mathcal{B}_{\text{Col } A} = \{(-3, -1, 2), (-1, 2, 5)\}$$

$$\dim \text{Col } A = 2 \quad (= \text{rang } A)$$

$$\text{Bemerk } \text{rang } A + \text{nullitet } A = 2 + 3 = 5$$

= 8 sager i A

## Koordinater

Vektorrum V med basis  $\mathcal{B} = (\bar{b}_1, \bar{b}_2, \dots, \bar{b}_n)$ ,  
der.  $\dim V = n$ .

$\forall v \in V \exists r_1, \dots, r_n : \bar{v} = r_1 \bar{b}_1 + r_2 \bar{b}_2 + \dots + r_n \bar{b}_n$ ,  
idet  $\bar{b}_1, \dots, \bar{b}_n$  lin. uafh. og  $\text{Sp} \{\bar{b}_1, \dots, \bar{b}_n\} = V$

Entydighed?

$$\text{antag } \bar{v} = r_1 \bar{b}_1 + r_2 \bar{b}_2 + \dots + r_n \bar{b}_n$$

$$\text{og } \bar{v} = s_1 \bar{b}_1 + s_2 \bar{b}_2 + \dots + s_n \bar{b}_n$$

$$\Rightarrow \bar{0} = (r_1 - s_1) \bar{b}_1 + (r_2 - s_2) \bar{b}_2 + \dots + (r_n - s_n) \bar{b}_n$$

$$\Rightarrow r_1 - s_1 = r_2 - s_2 = \dots = r_n - s_n = 0$$

$$\Rightarrow r_1 = s_1, r_2 = s_2, \dots, r_n = s_n$$

$$\Phi: V \rightarrow \mathbb{R}^n, \quad \Phi(\bar{v}) = [\bar{v}]_{\mathcal{B}} = (v_1, v_2, \dots, v_n)$$

$[\bar{v}]_{\mathcal{B}}$  koordinaterne til  $\bar{v}$  mht. basen  $\mathcal{B}$

Betravgt  $\bar{u}, \bar{v} \in V, r \in \mathbb{R}$

$$\bar{u} = v_1 \bar{b}_1 + \dots + v_n \bar{b}_n$$

$$\bar{v} = s_1 \bar{b}_1 + \dots + s_n \bar{b}_n$$

$$\bar{u} + \bar{v} = (v_1 + s_1) \bar{b}_1 + \dots + (v_n + s_n) \bar{b}_n$$

$$r\bar{u} = rv_1 \bar{b}_1 + \dots + rv_n \bar{b}_n$$

$$[\bar{u} + \bar{v}]_{\mathcal{B}} = (v_1 + s_1, \dots, v_n + s_n)$$

$$= (v_1, \dots, v_n) + (s_1, \dots, s_n)$$

$$= [\bar{u}]_{\mathcal{B}} + [\bar{v}]_{\mathcal{B}} \quad (1)$$

$$[r\bar{u}]_{\mathcal{B}} = (rv_1, \dots, rv_n) = r(v_1, \dots, v_n)$$

$$= r[\bar{u}]_{\mathcal{B}} \quad (2)$$

(1), (2)  $\Rightarrow \Phi$  er en lin. transf.

$V = \mathbb{R}^n, \mathcal{B} = E$  (den naturlige basis)

$$\forall \bar{v}: \bar{v} = v_1 \bar{e}_1 + v_2 \bar{e}_2 + \dots + v_n \bar{e}_n$$

$$\text{dvs. } [\bar{v}]_E = (v_1, v_2, \dots, v_n) = \bar{v}$$

$V$  med basis  $\mathcal{B} = (\bar{b}_1, \dots, \bar{b}_n)$

benævñk  $[\bar{b}_j]_{\mathcal{B}} = (0, \dots, 0, 1, 0, \dots, 0) = \bar{e}_j, \quad j = 1, \dots, n$

ekse.  $V = \text{Span}\{\bar{e}^x, \bar{e}^y\}, \mathcal{B} = (\bar{e}^x, \bar{e}^y)$

$$\cosh x = \frac{\bar{e}^x + \bar{e}^{-x}}{2} \in V, \quad \sinh x = \frac{\bar{e}^x - \bar{e}^{-x}}{2} \in V$$

$$[\cosh x]_{\mathcal{B}} = \left(\frac{1}{2}, \frac{1}{2}\right), \quad [\sinh x]_{\mathcal{B}} = \left(\frac{1}{2}, -\frac{1}{2}\right)$$

valg i stedet basen  $C = (\cosh x, \sinh x)$ :

$$[\bar{e}^*]_C = (1, 1), \quad [\bar{e}^{**}]_C = (1, -1)$$

ehs.  $T: \mathbb{R}^5 \rightarrow \mathbb{R}^3$ ,  $T(\bar{x}) = A\bar{x}$

$$A = \{\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4, \bar{a}_5\} = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

if. tilsligende udregning

$$T(\mathbb{R}^5) = \text{Col } A = \text{Sp}\{(-3, 1, 2), (-1, 2, 5)\}$$

$$\mathcal{B} = \mathcal{B}_{\text{Col } A} = ((-3, 1, 2), (-1, 2, 5))$$

Afslas på den reducede række-echelonform

$$[\bar{a}_1]_{\mathcal{B}} = (1, 0), \quad [\bar{a}_2]_{\mathcal{B}} = (-2, 0), \quad [\bar{a}_3]_{\mathcal{B}} = (0, 1)$$

$$[\bar{a}_4]_{\mathcal{B}} = (-1, 2), \quad [\bar{a}_5]_{\mathcal{B}} = (3, -2)$$

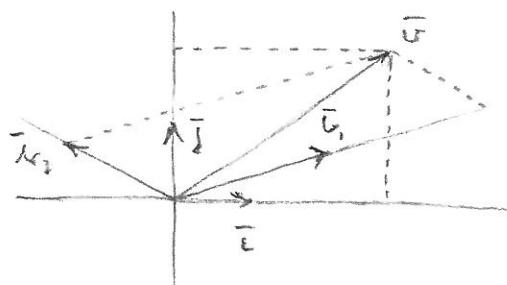
Besædne

$$[T(\bar{x})]_{\mathcal{B}} = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix} \bar{x} = B\bar{x},$$

hvor  $B$  er  $A$ 's reducede række-echelonform med omvejekter sluttet

Bazisskift fra  $\mathcal{E}$  til  $\mathcal{B}$  i  $\mathbb{R}^m$

ekse.



$$\bar{v} = [\bar{v}]_{\mathcal{E}} = (3, 2)$$

$$[\bar{v}]_{\mathcal{B}} = (2, 1)$$

ny basis  $\beta = (\bar{b}_1, \bar{b}_2, \dots, \bar{b}_n)$

$$\begin{aligned}\forall \bar{v} \in \mathbb{R}^m : \bar{v} &= r_1 \bar{b}_1 + r_2 \bar{b}_2 + \dots + r_n \bar{b}_n \\ &= [\bar{b}_1, \bar{b}_2, \dots, \bar{b}_n] \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}\end{aligned}$$

altså  $\bar{v} = B[v]_{\beta}$

$B = [\bar{b}_1, \bar{b}_2, \dots, \bar{b}_n]$  kaldes basisskiftematrix fra  $E$  til  $\beta$

$B$  er regular, da  $\bar{b}_1, \dots, \bar{b}_n$  lin. uafh.

Typisk er  $\bar{v}$  og  $B$  kendt, men  $[v]_{\beta}$  ukendt

Bemerk, at  $B[v]_{\beta} = \bar{v}$  svarer til et inhomogen ligningsystem.

eks.  $\bar{v} = (1, 4)$        $B = ((-1, 8), (1, -5))$

$$\left[ \begin{array}{cc|c} -1 & 1 & 1 \\ 1 & -5 & 4 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & -1 & -1 \\ 0 & 3 & -2 \end{array} \right] \sim \left[ \begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array} \right]$$

heraf  $[v]_{\beta} = (3, 2)$

Basisskift fra  $\beta$  til  $C$  i  $\mathbb{R}^m$

$\bar{v} = B[\bar{v}]_{\beta}$ ,  $B$  er basisskiftematrix fra  $E$  til  $\beta$

$\bar{v} = C[\bar{v}]_C$ ,  $C$  er basisskiftematrix fra  $E$  til  $C$

$$\Rightarrow B[\bar{v}]_{\beta} = C[\bar{v}]_C \Rightarrow [\bar{v}]_{\beta} = B^{-1}C[\bar{v}]_C$$

derfor  $B^{-1}C$  er basisskiftematrix fra  $\beta$  til  $C$

(alt.  $C^{-1}B$  er basisskiftematrix fra  $C$  til  $\beta$ )

ekn.  $[\bar{v}]_B = (5, 7)$        $B = ((-1, 8), (1, -5))$   
 $C = ((1, 4), (1, 1))$

Først at finde  $B^{-1}C$  kan vi løse matrix-ligningen  $BX = C$ :

$$\begin{bmatrix} -1 & 1 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 12 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 4 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}, \text{ der. } B^{-1}C = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$$

bestemmelser af  $[\bar{v}]_C$ :

$$\begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{der. } [\bar{v}]_C = (1, 1)$$

(alt. kunne vi have fundet  $C^{-1}B$  og udnyttet, at  $[\bar{v}]_C = C^{-1}B[\bar{v}]_B$ :

$$C^{-1}B = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}, [\bar{v}]_C = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}).$$

Basisændring fra  $B$  til  $C$ : vektorrummet  $V$

$$\bar{v} \in V, \quad B = (\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n)$$

$$C = (\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n)$$

Der gælder  $[\bar{v}]_B = S[\bar{v}]_C$ , hvor  $S$  er  
 basisændringsmatrix,  $S = ([\bar{v}_1]_B, [\bar{v}_2]_B, \dots, [\bar{v}_n]_B)$   
 (udledning i kompendium)

eksn.  $V = P_2$  (polynomier af højst 2. grad)

$$r(x) = 5x^2 + 6x + 2, \quad B = (1, x, x^2)$$

$$C = (x^2 - 2x, 3x^2 + 2x + 1, x^2 + 2x)$$

$$[\bar{v}]_B = (2, 6, 5)$$

$$[\bar{e}_1]_{\mathcal{B}} = (0, -2, 1), \quad [\bar{e}_2]_{\mathcal{B}} = (1, 2, 3), \quad [\bar{e}_3]_{\mathcal{B}} = (0, 2, 1)$$

$$\begin{array}{ccc} \left[ \begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ -2 & 2 & 2 & 6 \\ 1 & 3 & 1 & 5 \end{array} \right] & \sim & \left[ \begin{array}{ccc|c} 1 & 3 & 1 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 4 & 16 \end{array} \right] & \sim \left[ \begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{array} \right] \\ & \sim & \left[ \begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] & \sim \left[ \begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \end{array}$$

$$[r]_{\mathcal{E}} = (-1, 2, 0)$$

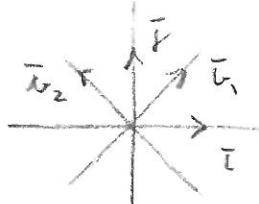
$$\begin{aligned} \text{kontrol: } & - (x^2 - 2x) + 2(3x^2 + 2x + 1) + 0(x^2 + 2x) \\ & = 5x^2 + 6x + 2 = p(x) \end{aligned}$$

geometrisk eks.

Koordinater mht.  $\mathcal{E} = (x, y)$

$$\text{Hvaad fremsstiller } 13x^2 - 10xy + 13y^2 - 72 = 0?$$

$$\text{Ny basis } \mathcal{B} = \left( \frac{\sqrt{2}}{2} \bar{e}_1 + \frac{\sqrt{2}}{2} \bar{e}_2, -\frac{\sqrt{2}}{2} \bar{e}_1 + \frac{\sqrt{2}}{2} \bar{e}_2 \right) = (\bar{b}_1, \bar{b}_2)$$



Koordinater mht.  $\mathcal{B} : (x_1, y_1)$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \mathcal{B} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2}x_1 - \frac{\sqrt{2}}{2}y_1 \\ \frac{\sqrt{2}}{2}x_1 + \frac{\sqrt{2}}{2}y_1 \end{bmatrix}$$

$$13x^2 - 10xy + 13y^2 - 72 = 0$$

$$13 \left( \frac{\sqrt{2}}{2}x_1 - \frac{\sqrt{2}}{2}y_1 \right)^2 - 10 \left( \frac{\sqrt{2}}{2}x_1 - \frac{\sqrt{2}}{2}y_1 \right) \left( \frac{\sqrt{2}}{2}x_1 + \frac{\sqrt{2}}{2}y_1 \right)$$

$$+ 13 \left( \frac{\sqrt{2}}{2}x_1 + \frac{\sqrt{2}}{2}y_1 \right)^2 - 72 = 0$$

$$\frac{1}{2} [13(x_1^2 - 2x_1y_1 + y_1^2) - 10(x_1^2 - y_1^2) + 3(x_1^2 + 2x_1y_1 - y_1^2)] = 72$$

$$\frac{1}{2} (16x_1^2 + 36y_1^2) = 72$$

$$8x_1^2 + 18y_1^2 = 72$$

$$\frac{x_1^2}{3^2} + \frac{y_1^2}{2^2} = 1$$

ellippe m. halvaksar 3 og 2

