

Orthogonalitet

Produktsymbol i \mathbb{R}^n , affabler $\mathbb{R}^m \times \mathbb{R}^n$ ind i \mathbb{R}

$$\bar{x} \cdot \bar{y} = \sum_i x_i y_i = \bar{x}^T \bar{y}$$

Bemerk $\bar{x} \cdot \bar{y} = \bar{y} \cdot \bar{x}$

$$\bar{x} \cdot \bar{x} \geq 0 \wedge \bar{x} \cdot \bar{x} = 0 \Leftrightarrow \bar{x} = \bar{0}$$

$$(\bar{x} + \bar{y}) \cdot \bar{z} = \bar{x} \cdot \bar{z} + \bar{y} \cdot \bar{z}$$

$$r\bar{x} \cdot \bar{y} = r(\bar{x} \cdot \bar{y}) = \bar{x} \cdot r\bar{y}$$

$$\text{Desuden } A\bar{x} \cdot \bar{y} = (A\bar{x})^T \bar{y} = \bar{x}^T A^T \bar{y} = \bar{x} \cdot (A^T \bar{y})$$

Norm i \mathbb{R}^n

$$\|\bar{x}\| = \sqrt{\bar{x} \cdot \bar{x}} = \sqrt{\sum_i x_i^2}$$

Bemerk $\|\bar{x}\| \geq 0 \wedge \|\bar{x}\| = 0 \Leftrightarrow \bar{x} = \bar{0}$

$$\|r\bar{x}\| = |r| \|\bar{x}\|$$

Metric i \mathbb{R}^n

$$d(\bar{x}, \bar{y}) = \|\bar{x} - \bar{y}\| = \sqrt{\sum_i (x_i - y_i)^2}$$

Orthogonale vektorer

$$\bar{x} \perp \bar{y} \Leftrightarrow \bar{x} \cdot \bar{y} = 0$$

Pythagoras

$$\begin{aligned} \|\bar{x} + \bar{y}\|^2 &= (\bar{x} + \bar{y}) \cdot (\bar{x} + \bar{y}) = (\bar{x} + \bar{y}) \cdot \bar{x} + (\bar{x} + \bar{y}) \cdot \bar{y} \\ &= \bar{x} \cdot \bar{x} + \bar{y} \cdot \bar{x} + \bar{x} \cdot \bar{y} + \bar{y} \cdot \bar{y} = \|\bar{x}\|^2 + 2\bar{x} \cdot \bar{y} + \|\bar{y}\|^2 \end{aligned}$$

$$\|\bar{x} + \bar{y}\|^2 = \|\bar{x}\|^2 + \|\bar{y}\|^2 \Leftrightarrow \bar{x} \cdot \bar{y} = 0$$

Orthogonal projektion af vektor på vektor



$$\bar{u} - c\bar{v} \perp \bar{v} \Leftrightarrow (\bar{u} - c\bar{v}) \cdot \bar{v} = 0$$

$$\Leftrightarrow \bar{u} \cdot \bar{v} - c\|\bar{v}\|^2 = 0 \Leftrightarrow c = \frac{\bar{u} \cdot \bar{v}}{\|\bar{v}\|^2}$$

$$\text{dvs. } \bar{u}_{\bar{v}} = \frac{\bar{u} \cdot \bar{v}}{\|\bar{v}\|^2} \bar{v}$$

Cauchy - Schwarz's ulighed

$$|\bar{x} \cdot \bar{y}| \leq \|\bar{x}\| \|\bar{y}\|$$

Bewis: (1) For $\bar{x} = \bar{0}$ v $\bar{y} = \bar{0}$ gælder lighedsstign

(2) $\bar{x}, \bar{y} \neq \bar{0}$: Betragt $\frac{\bar{x}}{\|\bar{x}\|}$ og $\frac{\bar{y}}{\|\bar{y}\|}$

$$0 \leq \left\| \frac{\bar{x}}{\|\bar{x}\|} + \frac{\bar{y}}{\|\bar{y}\|} \right\|^2 = \left(\frac{\bar{x}}{\|\bar{x}\|} + \frac{\bar{y}}{\|\bar{y}\|} \right) \cdot \left(\frac{\bar{x}}{\|\bar{x}\|} + \frac{\bar{y}}{\|\bar{y}\|} \right)$$

$$= 1 + \frac{\bar{x} \cdot \bar{y}}{\|\bar{x}\| \|\bar{y}\|} + 1 = 2 \left(1 + \frac{\bar{x} \cdot \bar{y}}{\|\bar{x}\| \|\bar{y}\|} \right)$$

$$\Leftrightarrow -1 \leq \frac{\bar{x} \cdot \bar{y}}{\|\bar{x}\| \|\bar{y}\|} \leq 1 \Leftrightarrow |\bar{x} \cdot \bar{y}| \leq \|\bar{x}\| \|\bar{y}\|$$

ekse. $\bar{a}, \bar{b} \in \mathbb{R}^3$, $|\bar{a} \cdot \bar{b}| \leq \|\bar{a}\| \|\bar{b}\|$

$$\Leftrightarrow |a_1 b_1 + a_2 b_2 + a_3 b_3| \leq \sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}$$

Trekantuligheden

$$\|\bar{x} + \bar{y}\| \leq \|\bar{x}\| + \|\bar{y}\|$$



Bewis: $\|\bar{x} + \bar{y}\|^2 = (\bar{x} + \bar{y}) \cdot (\bar{x} + \bar{y}) = \|\bar{x}\|^2 + 2\bar{x} \cdot \bar{y} + \|\bar{y}\|^2$
 $\leq \|\bar{x}\|^2 + 2\|\bar{x}\| \|\bar{y}\| + \|\bar{y}\|^2 = (\|\bar{x}\| + \|\bar{y}\|)^2$
 $\Leftrightarrow \|\bar{x} + \bar{y}\| \leq \|\bar{x}\| + \|\bar{y}\|$

Orthogonalt system af vektorer

ekse.: $(1, -2, 1), (0, 1, 2), (-5, -2, 1)$

Sætning: Et system af orthogonale vektorer, alle $\neq \bar{0}$, er lineart uafhængigt.

Bewis: $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k$ orthogonale, $\bar{v}_j \neq \bar{0}$, $j = 1, \dots, k$
 $r_1 \bar{v}_1 + r_2 \bar{v}_2 + \dots + r_k \bar{v}_k = \bar{0}$

Wirkprodukt med \bar{v}_j giver

$$0 + \dots + 0 + r_j \|\bar{v}_j\|^2 + 0 + \dots + 0 = 0, j = 1, \dots, k$$

$$\Rightarrow r_j = 0, j = 1, \dots, k$$

altså $\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k$ lineart uafh.

Ortonormalt system af vektorer

At normere en vektor, $\bar{v} \neq \bar{0}$, vil sige at multiplicere vektoren med skalaren $\frac{1}{\|\bar{v}\|}$. Bemærk $\left\| \frac{1}{\|\bar{v}\|} \bar{v} \right\| = 1$.

ekr. $\frac{1}{\sqrt{6}}(1, -2, 1)$, $\frac{1}{\sqrt{3}}(0, 1, 2)$, $\frac{1}{\sqrt{30}}(-5, -2, 1)$ er et ortonormalt system af vektorer.

Orthogonal basis

- når basisvektorene udgør et orthogonalsystem

ekr. $B = ((1, -2, 1), (0, 1, 2), (-5, -2, 1))$ er en orthogonal basis for \mathbb{R}^3 .

Koordinater mht. B

$$\bar{v} = r_1 \bar{b}_1 + r_2 \bar{b}_2 + \dots + r_n \bar{b}_n, \text{ dvs. } [\bar{v}]_B = (r_1, r_2, \dots, r_n)$$

Bestemmelse af r_j , $j = 1, \dots, n$

Prækprodukt mellem \bar{v} og \bar{b}_j

$$\bar{v} \cdot \bar{b}_j = 0 + \dots + 0 + r_j \|\bar{b}_j\|^2 + 0 + \dots + 0$$

$$\Rightarrow r_j = \frac{\bar{v} \cdot \bar{b}_j}{\|\bar{b}_j\|^2}, \quad j = 1, \dots, n$$

$$\bar{v} = \frac{\bar{v} \cdot \bar{b}_1}{\|\bar{b}_1\|^2} \bar{b}_1 + \frac{\bar{v} \cdot \bar{b}_2}{\|\bar{b}_2\|^2} \bar{b}_2 + \dots + \frac{\bar{v} \cdot \bar{b}_n}{\|\bar{b}_n\|^2} \bar{b}_n$$

ekr. fortsat

$$\bar{v} = (1, 4, 3)$$

$$[\bar{v}]_B = \left(-\frac{4}{5}, \frac{10}{5}, -\frac{10}{30} \right) = \left(-\frac{2}{5}, 2, -\frac{1}{3} \right)$$

Orthonormal basis

- når basisvektorene udgør et ortonormalsystem

$$C = (\bar{e}_1, \bar{e}_2, \dots, \bar{e}_n) \text{ orthonormal basis}$$

$$\bar{v} = (\bar{v} \cdot \bar{e}_1) \bar{e}_1 + (\bar{v} \cdot \bar{e}_2) \bar{e}_2 + \dots + (\bar{v} \cdot \bar{e}_n) \bar{e}_n$$

eks. Fortsat

$$C = \left(\frac{1}{\sqrt{3}}(1, -2, 1), \frac{1}{\sqrt{5}}(0, 1, 2), \frac{1}{\sqrt{30}}(-5, -2, 1) \right)$$

er en orthonormal basis for \mathbb{R}^3

$$\tilde{v} = (1, 4, 3)$$

$$[\tilde{v}]_C = \left(-\frac{4}{\sqrt{3}}, \frac{10}{\sqrt{5}}, -\frac{10}{\sqrt{30}} \right) = \left(-\frac{2}{3}\sqrt{6}, 2\sqrt{5}, -\frac{1}{3}\sqrt{30} \right)$$

Sætn. Et hvort system af lin. uafh. vektorer kan erstattes af et ortogonalt system, der har samme spænd.

Gram-Schmidt's ortogonaliseringsmetode:

$\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_k$ lin. uafh. vektorer

Sæt $\tilde{v}_1 = \tilde{u}_1$

$$\tilde{v}_2 = \tilde{u}_2 - \frac{\tilde{u}_2 \cdot \tilde{v}_1}{\|\tilde{v}_1\|^2} \tilde{v}_1$$

$$\tilde{v}_3 = \tilde{u}_3 - \frac{\tilde{u}_3 \cdot \tilde{v}_1}{\|\tilde{v}_1\|^2} \tilde{v}_1 - \frac{\tilde{u}_3 \cdot \tilde{v}_2}{\|\tilde{v}_2\|^2} \tilde{v}_2$$

:

$$\tilde{v}_k = \tilde{u}_k - \frac{\tilde{u}_k \cdot \tilde{v}_1}{\|\tilde{v}_1\|^2} \tilde{v}_1 - \dots - \frac{\tilde{u}_k \cdot \tilde{v}_{k-1}}{\|\tilde{v}_{k-1}\|^2} \tilde{v}_{k-1}$$

Bemerk $S_p(\tilde{v}_1, \dots, \tilde{v}_j) = S_p(\tilde{u}_1, \dots, \tilde{u}_j)$,

$j = 1, \dots, k$

Induktionsbevis for ortogonalitet:

(1) \tilde{v}_1 udgør et ortogonalsystem ($\tilde{u}_1 \neq 0$)

(2) antag, at $\tilde{v}_1, \dots, \tilde{v}_{i-1}$ udgør et ortogonalsystem

$$\begin{aligned} \tilde{v}_i \cdot \tilde{v}_j &= \tilde{u}_i \cdot \tilde{v}_j - 0 - \dots - 0 - \frac{\tilde{u}_i \cdot \tilde{v}_j}{\|\tilde{v}_j\|^2} \|\tilde{v}_j\|^2 - 0 - \dots - 0 \\ &= 0, \quad j = 1, \dots, i-1 \end{aligned}$$

$\Rightarrow \tilde{v}_1, \dots, \tilde{v}_i$ udgør et ortogonalsystem

gælder for $i = 2, \dots, k$

$\Rightarrow \tilde{v}_1, \dots, \tilde{v}_k$ udgør et ortogonalsystem

Efterfølgende normering af $\bar{v}_1, \dots, \bar{v}_k$ vil give et orthonormalt system af vektorer.

eks. $B = ((1,1,1), (0,1,1), (0,0,1))$ basis for \mathbb{R}^3

Bestem en ortogonal basis

$$\bar{e}_1 = (1,1,1)$$

$$\bar{e}_2 = (0,1,1) - \frac{2}{3}(1,1,1) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right), \bar{e}_2 = (-2,1,1)$$

$$\bar{e}_3 = (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1}{6}(-2,1,1) = \left(0, -\frac{1}{2}, \frac{1}{2}\right) \\ \bar{e}_3 = (0,-1,1)$$

$C = ((1,1,1), (-2,1,1), (0,-1,1))$ er en
ortogonal basis

En orthonormal basis:

$$\left(\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{6}}(-2,1,1), \frac{1}{\sqrt{2}}(0,-1,1)\right)$$

eks. Forstør $(1,-1,0,2), (1,1,1,2), (3,1,1,5)$ med
et orthonormalt system

Ford et ortogonalt system

$$\bar{v}_1 = (1,-1,0,2)$$

$$\bar{v}_2 = (1,1,1,2) - \frac{6}{7}(1,-1,0,2) = (0,2,1,1)$$

$$\begin{aligned} \bar{v}_3 &= (3,1,1,5) - \frac{12}{7}(1,-1,0,2) - \frac{8}{7}(0,2,1,1) \\ &= (1,3,1,1) - \frac{4}{5}(0,2,1,1) = \frac{1}{5}(3,1,-1,-1) \end{aligned}$$

ortogonalt system:

$$(1,-1,0,2), (0,2,1,1), (3,1,-1,-1)$$

orthonormalt system:

$$\frac{1}{\sqrt{6}}(1,-1,0,2), \frac{1}{\sqrt{6}}(0,2,1,1), \frac{1}{2\sqrt{3}}(3,1,-1,-1)$$

Indre produkt i V

$\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{R}$, hvor

$$\langle \bar{u}, \bar{v} \rangle = \langle \bar{v}, \bar{u} \rangle$$

$$\langle \bar{u}, \bar{v} \rangle \geq 0 \wedge \langle \bar{u}, \bar{v} \rangle = 0 \Leftrightarrow \bar{v} = \bar{0}$$

$$\langle \bar{u} + \bar{v}, \bar{w} \rangle = \langle \bar{u}, \bar{w} \rangle + \langle \bar{v}, \bar{w} \rangle$$

$$\langle c\bar{u}, \bar{v} \rangle = c \langle \bar{u}, \bar{v} \rangle = \langle u, c\bar{v} \rangle$$

eks. prisproduktet i \mathbb{R}^n er et indre produkt

eks. A m.m. er positiv definit

$$\Leftrightarrow \forall \bar{x} \in \mathbb{R}^n : \bar{x}^T A \bar{x} \geq 0 \wedge \bar{x}^T A \bar{x} = 0 \Leftrightarrow \bar{x} = \bar{0}$$

A positiv definit

$$\Rightarrow \langle \bar{x}, \bar{y} \rangle = \bar{x}^T A \bar{y} \text{ er et indre produkt i } \mathbb{R}^n$$

eks. $\langle f, g \rangle = \int_a^b f(x)g(x) dx$ er et indre produkt i P_n

Bemerk, at

norm, metrik, orthogonalitet, Pythagoras,
Cauchy-Schwarz, trekantuligheden,
Gram-Schmidt kan nem vides
generalisere til det generelle
indre produkt.

Indre produkt rum

er et vektorrum med et indre produkt
defineret

eks. P_2 med $\langle f, g \rangle = \int_0^1 f(x)g(x) dx$

Basis for $P_2 = (1, x, x^2)$

Bestem en orthonormal basis

Gram-Schmidt:

$$f_1(x) = 1$$

$$f_2(x) = x - \frac{\int_0^1 x \cdot 1 dx}{\int_0^1 1^2 dx} \cdot 1 = x - \frac{\frac{1}{2}}{1} \cdot 1 = x - \frac{1}{2}$$

$$f_3(x) = x^2 - \frac{\int_0^1 x^2 \cdot 1 dx}{\int_0^1 1^2 dx} \cdot 1 = \frac{\int_0^1 x^2 (x - \frac{1}{2}) dx}{\int_0^1 (x - \frac{1}{2})^2 dx} (x - \frac{1}{2})$$

$$= x^2 - \frac{\frac{1}{3}}{\frac{1}{3} - \frac{1}{2} + \frac{1}{4}} (x - \frac{1}{2}) = x^2 - \frac{1}{3} - \frac{1}{12} (x - \frac{1}{2})$$

$$= x^2 - x + \frac{1}{6}$$

Normierung:

$$\int_0^1 1^2 dx = 1$$

$$\int_0^1 (x - \frac{1}{2})^2 dx = \int_0^1 (x^2 - x + \frac{1}{4}) dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{1}{12}$$

$$\begin{aligned} \int_0^1 (x^2 - x + \frac{1}{6})^2 dx &= \int_0^1 (x^4 - 2x^3 + \frac{4}{3}x^2 - \frac{1}{3}x + \frac{1}{36}) dx \\ &= \frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36} = \frac{1}{180} \end{aligned}$$

Orthonormal basis:

$$(1, 2\sqrt{3}(x - \frac{1}{2}), 6\sqrt{5}(x^2 - x + \frac{1}{6}))$$

QR-Faktorisierung

$$A = [\bar{a}_1 \bar{a}_2 \dots \bar{a}_n] \quad m \times n, \quad \text{rang } A = n$$

Gram-Schmidt bei $\bar{a}_1, \bar{a}_2, \dots, \bar{a}_n$:

$$\begin{aligned} \bar{a}_1 &\in S_p\{\bar{v}_1\} &= S_p\{\bar{q}_1\} \\ \bar{a}_2 &\in S_p\{\bar{v}_1, \bar{v}_2\} &= S_p\{\bar{q}_1, \bar{q}_2\} \\ &\vdots \\ \bar{a}_n &\in S_p\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\} = S_p\{\bar{q}_1, \bar{q}_2, \dots, \bar{q}_n\} \end{aligned} \quad \left. \begin{array}{l} \bar{q}_j = \frac{\bar{v}_j}{\|\bar{v}_j\|}, \\ j=1, \dots, n \end{array} \right\}$$

$$\text{d.h. } \bar{a}_1 = r_{11} \bar{q}_1$$

$$\bar{a}_2 = r_{12} \bar{q}_1 + r_{22} \bar{q}_2$$

$$\vdots$$

$$\bar{a}_n = r_{1n} \bar{q}_1 + r_{2n} \bar{q}_2 + \dots + r_{nn} \bar{q}_n$$

$$\left. \begin{array}{l} r_{ij} = \bar{a}_j \cdot \bar{q}_i \\ j=1, \dots, n \\ i \leq j \end{array} \right\}$$

sat $Q = [\bar{q}_1 \bar{q}_2 \dots \bar{q}_n]$ $n \times n$

$$R = \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & r_{nn} \end{bmatrix}$$

$$A = QR$$

z.B. $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

$$G-S: \bar{v}_1 = (1, 1, 1, 1)$$

$$\bar{v}_2 = (2, 1, 0, 1) - \frac{1}{2}(1, 1, 1, 1) = (1, 0, -1, 0)$$

$$\bar{v}_3 = (1, 1, 2, 1) - \frac{5}{4}(1, 1, 1, 1) - \frac{1}{2}(1, 0, -1, 0)$$

$$= \left(\frac{1}{4}, -\frac{1}{4}, \frac{1}{4}, -\frac{1}{4}\right) \propto (1, -1, 1, -1)$$

$$\Rightarrow \bar{q}_1 = \frac{1}{2}(1, 1, 1, 1), \bar{q}_2 = \frac{1}{\sqrt{2}}(1, 0, -1, 0), \bar{q}_3 = \frac{1}{2}(1, -1, 1, -1)$$

$$r_{11} = \bar{q}_1 \cdot \bar{q}_1 = 2 \quad r_{12} = \bar{q}_2 \cdot \bar{q}_1 = 2 \quad r_{13} = \bar{q}_3 \cdot \bar{q}_1 = \frac{5}{2}$$

$$r_{22} = \bar{q}_2 \cdot \bar{q}_2 = \sqrt{2} \quad r_{23} = \bar{q}_3 \cdot \bar{q}_2 = -\frac{1}{2}$$

$$r_{33} = \bar{q}_3 \cdot \bar{q}_3 = \frac{1}{2}$$

$$A = QR = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 2 & \frac{5}{2} \\ 0 & \sqrt{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 1 & 1 & 1 \end{bmatrix}$$

Inhomogenes Ligningssystem

$$A\bar{x} = \bar{b}, A m \times n, \text{rang } A = n, A = QR$$

$$A\bar{x} = \bar{b} \Leftrightarrow QR\bar{x} = \bar{b} \Leftrightarrow Q^T QR\bar{x} = Q^T \bar{b} \Leftrightarrow R\bar{x} = Q^T \bar{b}$$

z.B. fortset

$$\bar{b} = (1, 3, 5, 3)$$

$$Q^T \bar{b} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 13 \\ \frac{15}{2} \\ -\frac{15}{2} \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 2 & -\frac{1}{2} & \frac{13}{2} \\ 0 & \sqrt{2} & -\frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array} \right]$$

$$\tilde{x} = (x, -z, 1)$$