

Orthogonalt Komplement

ehn.

$$\{\bar{v}\}^\perp = s_p\{\bar{v}\} = s_p\{(-2, 1)\}$$

$$(s_p\{\bar{v}\})^\perp = s_p\{(-2, 1)\}$$

$$(s_p\{(-2, 1)\})^\perp = ((s_p\{\bar{v}\})^\perp)^\perp = s_p\{\bar{v}\}$$

def vektorsystem S , $S \neq \emptyset$

$$S^\perp = \{\bar{v} \mid \bar{v} \cdot \bar{u} = 0 \text{ für alle } \bar{u} \in S\}$$

ehn. $S = \mathbb{R}^n \Rightarrow S^\perp = \{\bar{0}\}$

$$S = \{\bar{v}\} \Rightarrow S^\perp = \mathbb{R}^{n-1}$$

ehn. $S = \{\bar{u}\} = \{(1, 0, 2)\}$

$$S^\perp = \{\bar{v} \mid \bar{v} \cdot \bar{u} = 0\} = \text{Nul} [1 \ 0 \ 2]$$

$$= s_p \{(-2, 0, 1), (0, 1, 0)\}$$

Satz. S^\perp ist ein Unterraum

Beweis i. $\forall \bar{v} \in S: \bar{0} \cdot \bar{v} = 0 \Rightarrow \bar{0} \in S^\perp$ d.h. $S^\perp \neq \emptyset$

ii. $\bar{v}, \bar{w} \in S^\perp, \bar{u} \in S, r \in \mathbb{R}$

$$(\bar{v} + \bar{w}) \cdot \bar{u} = \bar{v} \cdot \bar{u} + \bar{w} \cdot \bar{u} = 0 + 0 = 0 \Rightarrow \bar{v} + \bar{w} \in S^\perp$$

$$(r\bar{v}) \cdot \bar{u} = r\bar{v} \cdot \bar{u} = r \cdot 0 = 0 \Rightarrow r\bar{v} \in S^\perp$$

i, ii $\Rightarrow S^\perp$ ist ein Unterraum

Satz. $S^\perp = (s_p S)^\perp$

Beweis i. $S = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n\}, \bar{w} \in s_p S$

$$\bar{w} = r_1 \bar{v}_1 + r_2 \bar{v}_2 + \dots + r_n \bar{v}_n$$

$$\bar{u} \in S^\perp \Rightarrow \bar{u} \cdot \bar{w} = 0 + 0 + \dots + 0 = 0$$

$$\Rightarrow \bar{u} \in (s_p S)^\perp, \text{ d.h. } S^\perp \subseteq (s_p S)^\perp$$

ii. $\bar{v} \in S \Rightarrow \bar{v} \in s_p S$

$$\bar{u} \in (s_p S)^\perp \Rightarrow \bar{u} \cdot \bar{v} = 0 \Rightarrow \bar{u} \in S^\perp, \text{ d.h. } (s_p S)^\perp \subseteq S^\perp$$

i, ii $\Rightarrow S^\perp = (s_p S)^\perp$

def $W \subseteq V$ underrum

$$W^\perp = \{ \bar{v} \mid \bar{v} \cdot \bar{w} \text{ for alle } \bar{w} \in W \}$$

ex. $W = \text{Sp} \{ (1, -1, -5, -1), (2, -1, -7, 0) \}$

$$W^\perp = \text{Nul} \begin{bmatrix} 1 & -1 & -5 & -1 \\ 2 & -1 & -7 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -5 & -1 \\ 2 & -1 & -7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -5 & -1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 3 & 2 \end{bmatrix}$$

$$W^\perp = \text{Sp} \{ (2, -3, 1, 0), (-1, -2, 0, 1) \}$$

Matrix A , $m \times n$

$$(\text{Row } A)^\perp = \text{Nul } A$$

$$(\text{Col } A)^\perp = \text{Nul } A^T \quad (\text{bemerk Row } A^T = \text{Col } A)$$

Orthogonal decomposition

$W \subseteq R^n$ underrum

Satz: $\forall \bar{v} \in R^n \exists \bar{w} \in W, \bar{z} \in W^\perp :$

$\bar{v} = \bar{w} + \bar{z}$ en entydig dekomposition

levis $(\bar{v}_1, \bar{v}_2, \dots, \bar{v}_n)$ orthonormal basis for W

sat $\bar{w} = (\bar{v} \cdot \bar{v}_1) \bar{v}_1 + (\bar{v} \cdot \bar{v}_2) \bar{v}_2 + \dots + (\bar{v} \cdot \bar{v}_k) \bar{v}_k \in W$

sat $\bar{z} = \bar{v} - \bar{w}$

$$\forall i : \bar{z} \cdot \bar{v}_i = (\bar{v} - \bar{w}) \cdot \bar{v}_i = \bar{v} \cdot \bar{v}_i - \bar{w} \cdot \bar{v}_i$$

$$= \bar{v} \cdot \bar{v}_i - \bar{v} \cdot \bar{v}_i = 0 \Rightarrow \bar{z} \in W^\perp$$

entydighet?

$$\text{antag } \bar{v} = \bar{w} + \bar{z} \wedge \bar{v} = \bar{w}' + \bar{z}',$$

$$\Rightarrow \bar{w} + \bar{z} = \bar{w}' + \bar{z}' \Leftrightarrow \bar{w} - \bar{w}' = \bar{z}' - \bar{z}$$

$$\text{bemerk } \bar{w} - \bar{w}' \in W \wedge \bar{z}' - \bar{z} \in W^\perp$$

$$\Rightarrow \bar{w} - \bar{w}' \in W \wedge \bar{w} - \bar{w}' \in W^\perp \Rightarrow \bar{w} - \bar{w}' = \bar{0}$$

$$\Rightarrow \bar{w} = \bar{w}' \Rightarrow \bar{z}' = \bar{z}$$

altsa entydig dekomposition

$$\text{Lös. } \bar{v} = (3, 1, 1) , \quad W = \text{Span}\{(2, -1, -2), (1, 1, -4)\}$$

orthogonal basis for W (G-S):

$$\bar{v}_1 = (2, -1, -2)$$

$$\bar{v}_2 = (1, 1, -4) - \frac{1}{2}(2, -1, -2) = (-1, 2, -2)$$

$$\text{orthonormal basis: } \left(\frac{1}{\sqrt{3}}(2, -1, -2), \frac{1}{\sqrt{3}}(-1, 2, -2) \right)$$

$$\bar{w} = \frac{1}{3}3 \cdot \frac{1}{\sqrt{3}}(2, -1, -2) + \frac{1}{3}(-3) \cdot \frac{1}{\sqrt{3}}(-1, 2, -2) = (1, -1, 0) \in W$$

$$\bar{z} = (3, 1, 1) - (1, -1, 0) = (2, 2, 1) \in W^\perp$$

Satz: $W \subseteq \mathbb{R}^n$ unendlich

$$\dim W + \dim W^\perp = n$$

Basis $\bar{v} \in \mathbb{R}^n \Rightarrow \exists \bar{w} \in W, \bar{z} \in W^\perp: \bar{v} = \bar{w} + \bar{z}$

basis for W : $\beta_1 = (\bar{b}_1, \dots, \bar{b}_k)$

basis for W^\perp : $\beta_2 = (\bar{c}_1, \dots, \bar{c}_m)$

$$\forall \bar{v} \in \mathbb{R}^n: \bar{v} = \bar{w} + \bar{z} = r_1 \bar{b}_1 + \dots + r_k \bar{b}_k + s_1 \bar{c}_1 + \dots + s_m \bar{c}_m \\ \Rightarrow \beta_1 \cup \beta_2 \text{ unabhängig } \mathbb{R}^n \quad (1)$$

$$\text{antag } \underbrace{r_1 \bar{b}_1 + \dots + r_k \bar{b}_k}_{\bar{w}} + \underbrace{s_1 \bar{c}_1 + \dots + s_m \bar{c}_m}_{\bar{z}} = \bar{0} \\ \Rightarrow \bar{w} + \bar{z} = \bar{0}$$

$\Rightarrow \bar{w} = \bar{0} \wedge \bar{z} = \bar{0}$, da einzigartig dekompr.

$\Rightarrow \bar{b}_1, \dots, \bar{b}_k, \bar{c}_1, \dots, \bar{c}_m$ lin. unabh. (2)

(1), (2) $\Rightarrow \beta_1 \cup \beta_2$ er Basis for \mathbb{R}^n

$$\Rightarrow k+m = n \Leftrightarrow \dim W + \dim W^\perp = n$$

Satz: $(W^\perp)^\perp = W$

Basis i: $\bar{w} \in W, \bar{v} \in W^\perp: \bar{w} \perp \bar{v} \Rightarrow \bar{v} \in (W^\perp)^\perp$
 $\Rightarrow (W^\perp)^\perp \supseteq W$

ii: $\dim W + \dim W^\perp = n$
 $\dim W^\perp + \dim (W^\perp)^\perp = n \quad \left\{ \Rightarrow \dim (W^\perp)^\perp = \dim W \right.$

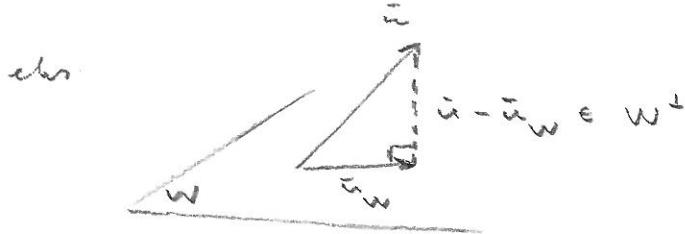
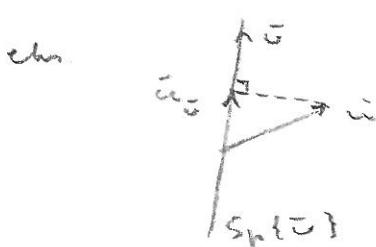
i, ii $\Rightarrow (W^\perp)^\perp = W$

Korollar $(S^\perp)^\perp = \text{Span } S$

Basis: $(S^\perp)^\perp = ((\text{Span } S)^\perp)^\perp = \text{Span } S$

Orthogonal projection på underrum

Def Den orthogonale projktion af \bar{u} på W
 er den entydigt bestemte vektor $\bar{w} \in W$,
 så $\bar{u} - \bar{w} \in W^\perp$



$$U_W : \mathbb{R}^n \rightarrow \mathbb{R}^n, U_W(\bar{u}) = \bar{w}$$

Sætn : U_W er en lineær operator

$$\text{bevis } U_W \bar{u}_1 = \bar{w}_1, U_W \bar{u}_2 = \bar{w}_2$$

$$\Rightarrow \exists \bar{z}_1, \bar{z}_2 \in W^\perp : \bar{u}_1 = \bar{w}_1 + \bar{z}_1, \bar{u}_2 = \bar{w}_2 + \bar{z}_2$$

(entydig dekomp.)

$$\Rightarrow \bar{u}_1 + \bar{u}_2 = (\bar{w}_1 + \bar{w}_2) + (\bar{z}_1 + \bar{z}_2), \bar{w}_1 + \bar{w}_2 \in W$$

$\bar{z}_1 + \bar{z}_2 \in W^\perp$

$$\Rightarrow U_W(\bar{u}_1 + \bar{u}_2) = \bar{w}_1 + \bar{w}_2 = U_W(\bar{u}_1) + U_W(\bar{u}_2) \quad (1)$$

$$U_W r\bar{u} = \bar{w}, r \in \mathbb{R}$$

$$\Rightarrow \exists \bar{z} : \bar{u} = \bar{w} + \bar{z} \text{ entydig dekomp.}$$

$$\Rightarrow r\bar{u} = r\bar{w} + r\bar{z}, r\bar{w} \in W, r\bar{z} \in W^\perp$$

$$\Rightarrow U_W(r\bar{u}) = r\bar{w} = rU_W(\bar{u}) \quad (2)$$

$$(1), (2) \Rightarrow U_W \text{ er lin.-op.}$$

Lemma C $n \times k$ matrix, $n \geq k$, rang $C = k$

$\Rightarrow C^T C$ er regular, $k \times k$

bevis antag $C^T C \bar{b} = \bar{0}$

$$\|C\bar{b}\|^2 = (C\bar{b})^T C\bar{b} = \bar{b}^T C^T C\bar{b} = \bar{b}^T \bar{0} = 0$$

$$\Rightarrow C\bar{b} = \bar{0} \Rightarrow \bar{b} = \bar{0}, \text{ altså}$$

$C^T C \bar{b} = \bar{0} \Leftrightarrow \bar{b} = \bar{0}$, heraf $C^T C$ regular

Matrixrepresentation af U_w

Sætn. $\beta = (\bar{e}_1, \dots, \bar{e}_k)$ basis for W

$$C = [\bar{e}_1 \dots \bar{e}_k], n \times k, W = \text{Col } C$$

$P_w = C(C^T C)^{-1} C^T$ repræsenterer U_w mht.

den naturlige basis for \mathbb{R}^n

bvis

$$\bar{u} \in \mathbb{R}^n, U_w(\bar{u}) = \bar{w}$$

sæt $\bar{v} = [\bar{w}]_{\beta}$, dvs. $\bar{w} = C\bar{v}$, jf. basisskift fra \bar{e} til β

$$\bar{u} - \bar{w} \in W^\perp = (\text{Col } C)^\perp = \text{Null } C^T$$

$$\Rightarrow C^T(\bar{u} - \bar{w}) = \bar{0} \Rightarrow C^T\bar{u} - C^T\bar{w} = \bar{0}$$

$$\Rightarrow C^T\bar{u} = C^T C \bar{v} = \bar{0} \Rightarrow C^T C \bar{v} = C^T \bar{u}$$

$$\Rightarrow \bar{v} = (C^T C)^{-1} C^T \bar{u}, \text{ jf. lemma}$$

$$U_w \bar{u} = \bar{w} = C\bar{v} = C(C^T C)^{-1} C^T \bar{u}$$

$$\text{herved } P_w = C(C^T C)^{-1} C^T$$

Bemerk P_w er symmetrisk og idempotent
(eftersv. opgave)

eks. fortæt $\bar{v} = (3, 1, 1)$ $W = \text{Span}\{(2, -1, -2), (1, 1, -4)\}$

$$C = \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ -2 & -4 \end{bmatrix}, \quad C^T C = \begin{bmatrix} 2 & -1 & -2 \\ 1 & 1 & -4 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} 9 & 9 \\ 9 & 18 \end{bmatrix}$$

$$C^T C = 9 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad (C^T C)^{-1} = \frac{1}{9} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$P_w = \frac{1}{9} \begin{bmatrix} 2 & 1 \\ -1 & 1 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & -2 \\ 1 & 1 & -4 \end{bmatrix}$$

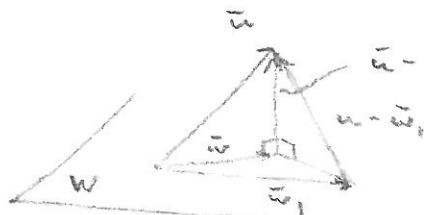
$$= \frac{1}{9} \begin{bmatrix} 3 & -1 \\ -3 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & -1 & -2 \\ 1 & 1 & -4 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \quad (\text{sym.})$$

$$P_w \bar{v} = \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 9 \\ -9 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{som tild.}$$

Kontrol for P_w idempotent:

$$\begin{aligned}
 P_w^2 &= \frac{1}{81} \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} = \frac{1}{81} \begin{bmatrix} 45 & -36 & -18 \\ -36 & 45 & -18 \\ -18 & -18 & 72 \end{bmatrix} \\
 &= \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} = P_w
 \end{aligned}$$

Mindste afstand fra \bar{v} til W



$$\begin{aligned}
 \|v\bar{v} - \bar{w}\|^2 &= \|v\bar{v} - \bar{w} + \bar{w} - \bar{w}\|^2 \\
 &= \|v\bar{v} - \bar{w}\|^2 + \|\bar{w} - \bar{w}\|^2 \quad (\text{Pythagoras}) \\
 &\geq \|v\bar{v} - \bar{w}\|^2
 \end{aligned}$$

$$\Rightarrow \|v\bar{v} - \bar{w}\| = \|v\bar{v} - P_w v\| = \|v\bar{v} - P_w v\| \quad \text{mindste afstand}$$

etrs. fortset

mindste afstand fra \bar{v} til W :

$$\|(3, 1, 1) - (1, -1, 0)\| = \|(2, 2, 1)\| = 3$$

Orthogonal projktion på det ortogonale komplement

$$\forall \bar{v} \in \mathbb{R}^3 : (I - P_w) \bar{v} = \bar{v} - P_w \bar{v} = \bar{v} - \bar{w} = \bar{z} \in W^\perp$$

dvs. $I - P_w$ repræsenterer U_{W^\perp} mht. ...

etrs. fortset

$$I - P_w = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{9} \begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$(I - P_w) \bar{v} = \frac{1}{9} \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 18 \\ 18 \\ 9 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad \text{som tidl.}$$

$$\begin{aligned}
 I - P_w \text{ er symmetrisk, da } (I - P_w)^T &= I^T - P_w^T = I - P_w, \\
 \text{og idempotent, da } (I - P_w)^2 &= (I - P_w)(I - P_w) \\
 &= I - P_w \cdot P_w + P_w^2 = I - P_w
 \end{aligned}$$

Egenverdioproblemet for P_w

$$P_w C = C(C^T C)^{-1} C^T C = C = \lambda C, \text{ dvs.}$$

vektorerne $\tilde{e}_1, \dots, \tilde{e}_k$ er lineært uafhængige egenvektorer hørende til egenverdiens λ .

Lad $(\tilde{e}_{k+1}, \dots, \tilde{e}_n)$ være basis for W^\perp

$$\text{Sæt } C_1 = [\tilde{e}_{k+1} \dots \tilde{e}_n], n \times (n-k)$$

$$P_w C_1 = C(C^T C)^{-1} C^T C_1 = 0 = 0 C_1, \text{ dvs.}$$

vektorerne $\tilde{e}_{k+1}, \dots, \tilde{e}_n$ er lineært uafhængige egenvektorer hørende til egenverdiens 0.

Altså

P_w har egenverdiens λ med algebrisk og geometrisk mult. k

P_w har egenverdiens 0 med algebrisk og geometrisk mult. $n-k$

W er egenrum hørende til egenverdiens λ

W^\perp er egenrum hørende til egenverdiens 0

P_w er invariant over for baseskift i W

Diagonalisering af P_w

$$B = [C \ C_1] \text{ diagonaliserer } P_w, \quad (B \text{ } n \times n)$$

$$B^{-1} P_w B = \begin{bmatrix} \lambda & \dots & 0 & \dots & 0 \end{bmatrix}$$

(udregning overflødig)