

Orthogonale matricer

Def. Q er orthogonal, når Q 's søjlevectorer er orthonormerede, dvs. $\bar{q}_i \cdot \bar{q}_j = \delta_{ij}$ (Kroneckers delta)
 (bemerk ulogisk betegnelse, men traditionel)

Sætning. Äkvivalente udsagn om Q

- i $Q^T Q = I$, jf. def. af Q
- ii Q er regular og $Q^{-1} = Q^T$, aflees af i
- iii $Q Q^T = I$, jf. regneregler for invers matrix

Orthogonale operatorer

Def T er orthogonal, når T repræsenteres af en orthogonal matrix, dvs. $T: \mathbb{R}^m \rightarrow \mathbb{R}^m$, $T(\bar{x}) = Q\bar{x}$

Sætning. Äkvivalente udsagn om T :

- i T er en orthogonal operator
- ii T er indre produkt bevarende
- iii T er normbevarende

$$\text{bewis } i \rightarrow ii \quad T(\bar{u}) \cdot T(\bar{v}) = Q\bar{u} \cdot Q\bar{v} = (Q\bar{u})^T Q\bar{v} \\ = \bar{u}^T Q^T Q\bar{v} = \bar{u}^T \bar{v} = \bar{u} \cdot \bar{v}$$

$$ii \rightarrow iii \quad \|T(\bar{v})\| = \|Q\bar{v}\| = \sqrt{Q\bar{v} \cdot Q\bar{v}} = \sqrt{\bar{v} \cdot \bar{v}} = \|\bar{v}\|$$

$$iii \Rightarrow i \quad \left. \begin{array}{l} \bar{q}_j = Q\bar{e}_j \\ \|Q\bar{e}_j\| = \|\bar{e}_j\| = 1 \end{array} \right\} \Rightarrow \|\bar{q}_j\| = 1 \quad (1)$$

$$\begin{aligned} \|\bar{q}_i + \bar{q}_j\|^2 &= \|Q\bar{e}_i + Q\bar{e}_j\|^2 = \|Q(\bar{e}_i + \bar{e}_j)\|^2 \\ &= \|\bar{e}_i + \bar{e}_j\|^2 = 1 = \|\bar{q}_i\|^2 + \|\bar{q}_j\|^2 \\ \Rightarrow \bar{q}_i &\perp \bar{q}_j \text{ (Pythagoras)} \quad (2) \end{aligned}$$

$$(1), (2) \Rightarrow \bar{q}_i \cdot \bar{q}_j = \delta_{ij}$$

Satz. En ortogonal operator er vinkelbevarende
 hvis vinkel mellem \bar{u} og \bar{v} : $\alpha \in [0; \pi]$
 vinkel mellem $T\bar{u}$ og $T\bar{v}$: $\beta \in [0, \pi]$

$$\begin{aligned} T(\bar{u}) \cdot T(\bar{v}) = \bar{u} \cdot \bar{v} &\Leftrightarrow \|T(\bar{u})\| \|T(\bar{v})\| \cos \beta = \|\bar{u}\| \|\bar{v}\| \cos \alpha \\ &\Leftrightarrow \|\bar{u}\| \|\bar{v}\| \cos \beta = \|\bar{u}\| \|\bar{v}\| \cos \alpha \\ &\Leftrightarrow \cos \beta = \cos \alpha \\ &\Leftrightarrow \alpha = \beta, \text{ m\u00e5r } \alpha, \beta \in [0, \pi] \end{aligned}$$

Orthogonale matricer fortset

Sætning a $\det Q = \pm 1$

b Q_1 og Q_2 orthogonale $\Rightarrow Q_1 Q_2$ orthogonal

$$\begin{aligned} \text{Væri a } Q^T Q &= I \Rightarrow \det Q^T \det Q = \det I \\ &\Rightarrow (\det Q)^2 = 1 \quad (\det Q^T = \det Q) \\ &\Rightarrow \det Q = \pm 1 \end{aligned}$$

$$\begin{aligned} b \quad (Q_1 Q_2)^T &= Q_2^T Q_1^T = Q_2^{-1} Q_1^{-1} = (Q_1 Q_2)^{-1} \\ &\Rightarrow (Q_1 Q_2)^T (Q_1 Q_2) = I \Rightarrow Q_1 Q_2 \text{ orthogonal} \end{aligned}$$

der vi har orthogonale matricer

$$i \quad \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$ii \quad \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix} \quad (3 \times 3 \text{ Helmertmatrix})$$

$$iii \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & -\frac{2}{\sqrt{12}} \end{bmatrix} \quad (4 \times 4 \text{ Helmertmatrix})$$

Orthogonale operatorer fortset

Sætning a. T orthogonal $\Rightarrow T^{-1}$ orthogonal

b. T, U orthogonale $\Rightarrow TU$ orthogonale
(eftersvarende i opgave)

eksempel. $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $T(\bar{x}) = Q\bar{x}$, Q orthogonal

$$\text{Bestem } Q, \text{ så } T \underbrace{\left(\frac{1}{\sqrt{10}} (3, 1, 0) \right)}_{\bar{v}} = \underbrace{\frac{1}{\sqrt{5}} (0, -2, 1)}_{\bar{w}}$$

Vælg orthogonale matricer P og R , fx så

$$P\bar{v} = \bar{e}_1 \quad \wedge \quad R\bar{w} = \bar{e}_1 \quad \Rightarrow \quad R^T P \bar{v} = \bar{w}$$

$$\Rightarrow Q\bar{v} = \bar{w}, \quad Q \text{ orthogonal}$$

$$P\bar{v} = \bar{e}_1 \Rightarrow \bar{v} = P^T \bar{e}_1 \Rightarrow \bar{v} = (P^T)_1$$

$$\Rightarrow (P^T)_2, (P^T)_3 \in \text{Nul } [3 \mid 0]$$

$$\Rightarrow \begin{cases} (P^T)_2 = \frac{1}{\sqrt{10}} (-1, 3, 0) \\ (P^T)_3 = (0, 0, 1) \end{cases} \quad (\text{fx})$$

$$P^T = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & -1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & \sqrt{10} \end{bmatrix} \Rightarrow P = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & 1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & \sqrt{10} \end{bmatrix}$$

$$R\bar{w} = \bar{e}_1 \Rightarrow \bar{w} = R^T \bar{e}_1 \Rightarrow \bar{w} = (R^T)_1$$

$$\Rightarrow (R^T)_2, (R^T)_3 \in \text{Nul } [0 \mid -2 \mid 1]$$

$$(R^T)_2 = (1, 0, 0)$$

$$(R^T)_3 = \frac{1}{\sqrt{5}} (0, 1, 2) \quad (\text{fx})$$

$$R^T = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 & \sqrt{5} & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$Q = R^T P = \frac{1}{\sqrt{5}} \frac{1}{\sqrt{10}} \begin{bmatrix} 0 & \sqrt{5} & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & \sqrt{10} \end{bmatrix}$$

$$= \frac{1}{5\sqrt{2}} \begin{bmatrix} -\sqrt{5} & 3\sqrt{5} & 0 \\ -6 & -2 & \sqrt{10} \\ 3 & 1 & 2\sqrt{10} \end{bmatrix}$$

kontrol

$$Q \tilde{w} = \frac{1}{5\sqrt{2}} \frac{1}{\sqrt{10}} \begin{bmatrix} -\sqrt{5} & 3\sqrt{5} & 0 \\ -6 & -2 & \sqrt{10} \\ 3 & 1 & 2\sqrt{10} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{10\sqrt{5}} \begin{bmatrix} 0 \\ -20 \\ 10 \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} (0, -2, 1) = \tilde{w}$$

Orthogonale operationer i \mathbb{R}^2

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2, T(\tilde{x}) = Q \tilde{x}, Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ orthogonal}$$

$$\|(a, c)\| = 1 \Rightarrow a^2 + c^2 = 1 \Rightarrow \exists \theta : a = \cos \theta, c = \sin \theta$$

$$(a, c) \cdot (b, d) = 0 \Rightarrow ab + cd = 0 \Rightarrow b \cos \theta + d \sin \theta = 0$$

$$\Rightarrow (b = -\sin \theta \wedge d = \cos \theta) \vee (b = \sin \theta \wedge d = -\cos \theta)$$

(1) $Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \det Q = 1$

T er en drejning m. drejningsvinkel θ ,

if. tidi. eks.

(2) $Q = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}, \det Q = -1$

eigenverdier og egenvektorer:

$$\begin{vmatrix} \cos \theta - \lambda & \sin \theta \\ \sin \theta & -\cos \theta - \lambda \end{vmatrix} = -(\cos \theta - \lambda)(\cos \theta + \lambda) - \sin^2 \theta$$

$$= -(\cos^2 \theta - \lambda^2) - \sin^2 \theta = \lambda^2 - 1 = 0 \text{ for } \lambda = \pm 1$$

$$\lambda = 1: \begin{bmatrix} \cos \theta - 1 & \sin \theta \\ \sin \theta & -\cos \theta - 1 \end{bmatrix} \sim \begin{bmatrix} -\tan \frac{\theta}{2} & 1 \\ \tan \frac{\theta}{2} + 1 & 0 \end{bmatrix} \sim \begin{bmatrix} \tan \frac{\theta}{2} & -1 \\ 0 & 0 \end{bmatrix}$$

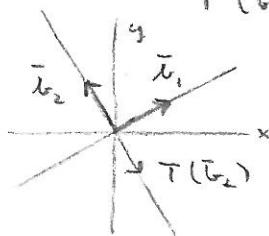
$$\tilde{v}_1 = r (1, \tan \frac{\theta}{2}), r \neq 0$$

$$T(\tilde{v}_1) = 1 \tilde{v}_1 = \tilde{v}_1$$

$$\pi = -1 : \begin{bmatrix} \cos \theta + 1 & \sin \theta \\ \sin \theta & -\cos \theta + 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \tan \frac{\theta}{2} \\ 1 & \tan \frac{\theta}{2} \end{bmatrix} \sim \begin{bmatrix} 1 & \tan \frac{\theta}{2} \\ 0 & 0 \end{bmatrix}$$

$$\bar{b}_2 = s(-\tan \frac{\theta}{2}, 1), s \neq 0$$

$$T(\bar{b}_2) = -1 \bar{b}_2 = -\bar{b}_2$$



$$\bar{b}_1 \cdot \bar{b}_2 = -\tan \frac{\theta}{2} + \tan \frac{\theta}{2} = 0 \\ \Rightarrow \bar{b}_1 \perp \bar{b}_2$$

T er en spejling om

$$\text{linien } y = \tan \frac{\theta}{2} x$$

ehn. $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2, T(\bar{x}) = \frac{1}{5} \begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix} \bar{x} = Q\bar{x}$

spejling, da $\det Q = -1$

spejlingsretse $y = \tan \frac{\theta}{2} x, \tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} = \frac{\frac{4}{5}}{1 + \frac{3}{5}} = \frac{1}{2}$
 $y = \frac{1}{2} x$

ehn. $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2, T(\bar{x}) = \frac{1}{5} \begin{bmatrix} -3 & 4 \\ -4 & -3 \end{bmatrix} \bar{x} = Q\bar{x}$

drejning, da $\det Q = 1$

drejningsvinkel $\theta = \arctan \frac{4}{3} + \pi \approx 233,1^\circ$

Sætning. T, U ortogonale operatoren i \mathbb{R}^2

a. T, U spejlinger $\Rightarrow TU$ er en drejning

b. T drejning
 U spejling } $\Rightarrow TU$ og UT er spejlinger

Basis a. $\det T = -1, \det U = -1 \Rightarrow \det TU = (-1)^2 = 1$

$\Rightarrow TU$ er en drejning

b. $\det T = -1, \det U = 1 \Rightarrow \det TU = \det UT = -1$

$\Rightarrow TU$ og UT er spejlinger

Flytninger

Def. $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$ er en flytning, hvis $\forall \bar{u}, \bar{v}:$

$$\|F(\bar{u}) - F(\bar{v})\| = \|\bar{u} - \bar{v}\|, \text{ dvs. afstandsbewarende}$$

Bemerk en ortogonal operator er en flytning,
idet $\|T(\bar{u}) - T(\bar{v})\| = \|T(\bar{u} - \bar{v})\| = \|\bar{u} - \bar{v}\|$

Bemerk en flytning, der er lineær, er en ort. op.,
idet $\|F(\bar{v})\| = \|F(\bar{u}) - \bar{v}\| = \|F(\bar{v}) - F(\bar{u})\|$
 $= \|\bar{v} - \bar{u}\| = \|\bar{v}\|$

Def. $F_{\bar{v}}: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $F_{\bar{v}}(\bar{u}) = \bar{u} + \bar{v}$ er en translation
med vektoren \bar{v}

Bemerk $F_{\bar{v}}(\bar{v}) = \bar{v}$, dvs. for $\bar{v} \neq \bar{0}$ er $F_{\bar{v}}$ ikke lineær

Bemerk $\|F_{\bar{v}}(\bar{u}) - F_{\bar{v}}(\bar{v})\| = \|\bar{u} + \bar{v} - (\bar{v} + \bar{v})\| = \|\bar{u} - \bar{v}\|$
dvs. $F_{\bar{v}}$ er en flytning

Sætn. $F_{\bar{v}}$ og $F_{\bar{z}}$ flytninger $\Rightarrow F_{\bar{v}}F_{\bar{z}}$ er en flytning

bevis $\|F_{\bar{v}}F_{\bar{z}}(\bar{u}) - F_{\bar{v}}F_{\bar{z}}(\bar{v})\| = \|F_{\bar{z}}(\bar{u}) - F_{\bar{z}}(\bar{v})\| = \|\bar{u} - \bar{v}\|$

Lemma En flytning $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ med $T(\bar{0}) = \bar{0}$
er en ortogonal operator

bevis i: $\|T(\bar{u})\| = \|T(\bar{u}) - T(\bar{0})\| = \|\bar{u} - \bar{0}\| = \|\bar{u}\|$

$$\text{ii: } \begin{aligned} \|T(\bar{u}) - T(\bar{v})\|^2 &= \|T(\bar{u})\|^2 - 2T(\bar{u}) \cdot T(\bar{v}) + \|T(\bar{v})\|^2 \\ &= \|\bar{u}\|^2 - 2T(\bar{u}) \cdot T(\bar{v}) + \|\bar{v}\|^2 \quad (1) \end{aligned}$$

$$\begin{aligned} \|T(\bar{u}) - T(\bar{v})\|^2 &= \|\bar{u} - \bar{v}\|^2 \quad (T \text{ er en flytning}) \\ &= \|\bar{u}\|^2 - 2\bar{u} \cdot \bar{v} + \|\bar{v}\|^2 \quad (2) \end{aligned}$$

$$(1), (2) \Rightarrow T(\bar{u}) \cdot T(\bar{v}) = \bar{u} \cdot \bar{v}$$

$$\begin{aligned}
 \text{(iii)} \quad & \|T(\bar{u} + \bar{v}) - T(\bar{u}) - T(\bar{v})\|^2 \\
 &= \|T(\bar{u} + \bar{v})\|^2 + \|T(\bar{v})\|^2 + \|T(\bar{u})\|^2 \\
 &\quad - 2 T(\bar{u} + \bar{v}) \cdot T(\bar{u}) - 2 T(\bar{u} + \bar{v}) \cdot T(\bar{v}) - 2 T(\bar{u}) \cdot T(\bar{v}) \\
 &= \|\bar{u} + \bar{v}\|^2 + \|\bar{v}\|^2 + \|\bar{u}\|^2 \\
 &\quad - 2(\bar{u} + \bar{v}) \cdot \bar{u} - 2(\bar{u} + \bar{v}) \cdot \bar{v} - 2\bar{u} \cdot \bar{v} \\
 &= \|\bar{u}\|^2 + 2\bar{u} \cdot \bar{v} + \|\bar{v}\|^2 + \|\bar{u}\|^2 + \|\bar{v}\|^2 \\
 &\quad - 2\|\bar{u}\|^2 - 2\bar{u} \cdot \bar{v} - 2\bar{u} \cdot \bar{v} - 2\|\bar{v}\|^2 + 2\bar{u} \cdot \bar{v} \\
 &= 0 \\
 \Rightarrow \quad & T(\bar{u} + \bar{v}) = T(\bar{u}) + T(\bar{v}) \quad (1)
 \end{aligned}$$

$$\begin{aligned}
 & \|T(c\bar{u}) - cT(\bar{u})\|^2 \\
 &= \|T(\bar{u})\|^2 - 2c T(c\bar{u}) \cdot T(\bar{u}) + c^2 \|T(\bar{u})\|^2 \\
 &= \|c\bar{u}\|^2 - 2c(c\bar{u}) \cdot \bar{u} + c^2 \|\bar{u}\|^2 \\
 &= c^2 \|\bar{u}\|^2 - 2c^2 \|\bar{u}\|^2 + c^2 \|\bar{u}\|^2 \\
 &= 0 \\
 \Rightarrow \quad & T(c\bar{u}) = cT(\bar{u}) \quad (2)
 \end{aligned}$$

(1), (2) $\Rightarrow T$ er linear

iv en normbevarende linear operator
er en orthogonal operator.

Sætn. F vilk. flytning

$F = F_{\bar{v}} T$, hvor T er en orthogonal op.
og $F_{\bar{v}}$ er en translation

bewis Sat $T(\bar{v}) = F(\bar{v}) - F(\bar{o})$

$T(\bar{o}) = F(\bar{o}) - F(\bar{o}) = \bar{o} \Rightarrow T$ er en lin. op.

Sat $F_{\bar{v}}(\bar{v}) = F(\bar{v}) = \bar{v}$

$F(\bar{v}) = T(\bar{v}) + \bar{v} = F_{\bar{v}} T(\bar{v})$

Enhver flytning kan sammensættes af en
orthogonal transformation efterfulgt af en
translation.

$$\text{dvs } F: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad F(1,2) = (5,3)$$

$$F(3,1) = (3,4)$$

$$F(-2,1) = (-6,0)$$

$$F(\bar{v}) = T(\bar{v}) + \bar{t}, \quad T(\bar{v}) = Q\bar{v}, \quad Q \text{ orthogonal}$$

$$\begin{aligned} T(1,2) + \bar{t} &= (5,3) \\ T(3,1) + \bar{t} &= (3,4) \end{aligned} \quad \left. \begin{array}{l} \Rightarrow T(-2,1) = (2,-1) \\ \Rightarrow T(5,0) = (-3,4) \end{array} \right\} =$$

$$\begin{aligned} T(3,1) + \bar{t} &= (3,4) \\ T(-2,1) + \bar{t} &= (-6,0) \end{aligned} \quad \left. \begin{array}{l} \Rightarrow T(5,0) = (-3,4) \end{array} \right\}$$

$$Q \begin{bmatrix} -2 & 5 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & 1 \\ 5 & 0 \end{bmatrix} Q^T = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

matrixvergelijkingen lossen met Q^T :

$$\begin{bmatrix} -2 & 1 & | & 2 & -1 \\ 5 & 0 & | & -3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & | & 1 & 2 \\ -2 & 1 & | & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & | & 1 & 2 \\ 0 & 3 & | & 4 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 5 & 10 & | & 5 & 10 \\ 0 & 5 & | & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 5 & 0 & | & -3 & 4 \\ 0 & 5 & | & 4 & 3 \end{bmatrix}$$

$$\Rightarrow Q^T = \frac{1}{5} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix}, \quad Q = Q^T \quad (\text{Q}^T \text{ symm.}) \quad *$$

$$\bar{t} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} - \frac{1}{5} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

T is een spiegeling, da $\det Q = -1$

spiegelsaks:

$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} = \frac{\frac{4}{5}}{1 - \frac{3}{5}} = 2, \quad \text{dvs } y = 2x$$

F is een spiegeling om lijnen $y = 2x$

afgevolgt of een translatie met
vectoren $(4,1)$

* alternatieve bestemming van Q :

$$\begin{aligned} Q &= \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 5 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \cdot \frac{1}{5} \begin{bmatrix} 0 & -5 \\ -1 & 2 \end{bmatrix} \\ &= \frac{1}{5} \begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 1 & 2 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} -3 & 4 \\ 4 & 3 \end{bmatrix} \end{aligned}$$