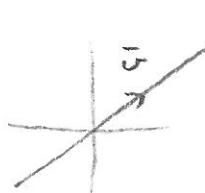


Span af vektorer

Betrægt et system af vektorer $\bar{v}_1, \dots, \bar{v}_n \in \mathbb{R}^m$

$$Sp\{\bar{v}_1, \dots, \bar{v}_n\} = \{r_1\bar{v}_1 + \dots + r_n\bar{v}_n \mid r_1, \dots, r_n \in \mathbb{R}\}$$

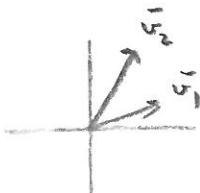
eks. i \mathbb{R}^2



$$Sp\{\bar{v}\}$$



$$Sp\{\bar{v}\} = \{0\}$$



$$Sp\{\bar{v}_1, \bar{v}_2\} = \mathbb{R}^2$$



$$\begin{aligned} Sp\{\bar{v}_1, \bar{v}_2\} &= Sp\{\bar{v}_1\} \\ &= Sp\{\bar{v}_2\} \end{aligned}$$

Bemerk $\bar{v} \in Sp\{\bar{v}_1, \dots, \bar{v}_n\}$

$$\Leftrightarrow \exists \bar{r} \in \mathbb{R}^k : A\bar{r} = \bar{v}, \text{ hvor } A = [\bar{v}_1 \dots \bar{v}_n]$$

dvs. \bar{v} er lin. komb. af A's spfier

eks. $\bar{v}_1 = (1, 0, -2)$

$$\bar{v}_2 = (-1, 3, 2)$$

$$\bar{v}_3 = (2, 5, -4)$$

$$\bar{v} = (3, -7, -3)$$

$$\bar{v} \in Sp\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}?$$

$$\left[\begin{array}{ccc|c} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & 0 & 0 & 3 \end{array} \right], \quad \text{mij}$$

eks. $\bar{v}_1 = (2, -1, 1)$

$$\bar{v}_2 = (0, 3, -2)$$

$$\bar{v}_3 = (6, 5, 1)$$

$$\bar{v} = (10, 3, 3)$$

$$\bar{v} \in Sp\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}?$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 6 & 10 \\ -1 & 3 & 5 & 3 \\ 1 & -2 & 1 & 3 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 3 & 8 & 8 \\ 0 & -2 & -2 & -2 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right], \quad \text{je}$$

$$\left(\bar{v} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}, r \in \mathbb{R} \right) \quad \text{fx } \bar{v} = (5, 1, 0)$$

Ekvalente udsagn om $A \in \mathbb{R}^{m \times n}$ ①

a $\text{Sp}\{\bar{a}_1, \dots, \bar{a}_n\} = \mathbb{R}^m$

b $\forall \bar{b} \in \mathbb{R}^m : A\bar{x} = \bar{b}$ er konsistent

c $\text{rang } A = m$

d Ingen multæller i en med A ekvivalent række-chelonform

e Pivotelement i alle rækker i en med A ekvivalent række-chelonform

Beweis for b \Leftrightarrow c:

$\Rightarrow : A \sim R$, hvor R er række-chelonform

$$\Rightarrow \exists \bar{c} : [R | \bar{c}_m] \sim [A | \bar{c}]$$

$A\bar{x} = \bar{c}$ konsistent $\Rightarrow R$ har ingen multæller
 $\Rightarrow \text{rang } A = m$

$$\Leftarrow : [A | \bar{c}] \sim [R | \bar{c}]$$

$\text{rang } A = m \Rightarrow R$ har ingen multæller
 $\Rightarrow A\bar{x} = \bar{c}$ er konsistent

Sætn.

$$\text{Sp}\{\bar{v}_1, \dots, \bar{v}_k, \bar{v}\} = \text{Sp}\{\bar{v}_1, \dots, \bar{v}_k\}$$

$$\Leftrightarrow \bar{v} \in \text{Sp}\{\bar{v}_1, \dots, \bar{v}_k\}$$

Beweis $\Leftarrow : \bar{v} \in \text{Sp}\{\bar{v}_1, \dots, \bar{v}_k\} \Rightarrow \exists r_1, \dots, r_k : \bar{v} = r_1\bar{v}_1 + \dots + r_k\bar{v}_k$

Betrægt $\bar{w} \in \text{Sp}\{\bar{v}_1, \dots, \bar{v}_k, \bar{v}\}$

$$\Rightarrow \exists s_1, \dots, s_k, s : \bar{w} = s_1\bar{v}_1 + \dots + s_k\bar{v}_k + s\bar{v}$$

$$\Rightarrow \bar{w} = s_1\bar{v}_1 + \dots + s_k\bar{v}_k + s(r_1\bar{v}_1 + \dots + r_k\bar{v}_k)$$

$$= (s_1 + sr_1)\bar{v}_1 + \dots + (s_k + sr_k)\bar{v}_k$$

$$\Rightarrow \text{Sp}\{\bar{v}_1, \dots, \bar{v}_k, \bar{v}\} \subseteq \text{Sp}\{\bar{v}_1, \dots, \bar{v}_k\} \quad (1)$$

fortsættes

Detrage $\bar{w} \in S_p\{\bar{v}_1, \dots, \bar{v}_n\}$

$$\Rightarrow \exists r_1, \dots, r_k : \bar{w} = r_1 \bar{v}_1 + \dots + r_k \bar{v}_k$$

$$= r_1 \bar{v}_1 + \dots + r_n \bar{v}_n + 0 \bar{v}$$

$$\Rightarrow \bar{w} \in S_p\{\bar{v}_1, \dots, \bar{v}_k, \bar{v}\}$$

$$\Rightarrow S_p\{\bar{v}_1, \dots, \bar{v}_k\} \subseteq S_p\{\bar{v}_1, \dots, \bar{v}_n, \bar{v}\} \quad (2)$$

$$(1), (2) \Rightarrow S_p\{\bar{v}_1, \dots, \bar{v}_k, \bar{v}\} = S_p\{\bar{v}_1, \dots, \bar{v}_k\}$$

\Rightarrow : indirekt, antag $\bar{v} \notin S_p\{\bar{v}_1, \dots, \bar{v}_k\}$

$$\text{benark } \bar{v} = 0 \bar{v}_1 + \dots + 0 \bar{v}_n + 1 \bar{v}$$

$$\Rightarrow \bar{v} \in S_p\{\bar{v}_1, \dots, \bar{v}_k, \bar{v}\}$$

$$\Rightarrow S_p\{\bar{v}_1, \dots, \bar{v}_k, \bar{v}\} \neq S_p\{\bar{v}_1, \dots, \bar{v}_k\}$$

modstrid, alltså $\bar{v} \in S_p\{\bar{v}_1, \dots, \bar{v}_k\}$

ekr $S_p\{(1,0,0), (1,1,0), (1,1,1), (0,0,1)\}$

$$= S_p\{(1,0,0), (1,1,0), (0,0,1)\}$$

$$= S_p\{(1,0,0), (0,1,0), (0,0,1)\} = \mathbb{R}^3$$

Linear abhängig / linear unabhängig

Def. $\bar{v}_1, \dots, \bar{v}_n$ er lineært uafhængigt, når

$$r_1\bar{v}_1 + \dots + r_n\bar{v}_n = \bar{0} \Rightarrow r_1 = \dots = r_n = 0$$

Def. $\bar{v}_1, \dots, \bar{v}_n$ er lineært afhængigt, når

$$\exists (r_1, \dots, r_n) \neq (0, \dots, 0) : r_1\bar{v}_1 + \dots + r_n\bar{v}_n = \bar{0}$$

Bemerk

$\bar{v}_1, \dots, \bar{v}_n$ lin. uafh. $\Leftrightarrow A\bar{v} = \bar{0}$ har kun null løsning.

$\bar{v}_1, \dots, \bar{v}_n$ lin. afh. $\Leftrightarrow A\bar{v} = \bar{0}$ har egentlige løsninger

eks. Er $\bar{v}_1 = (1, 1, 0)$, $\bar{v}_2 = (3, -1, 4)$, $\bar{v}_3 = (-1, 2, -3)$ lin. uafh.?

$$\begin{bmatrix} 1 & 3 & -1 \\ 1 & -1 & 2 \\ 0 & 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & -4 & 3 \\ 0 & 4 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix}, \text{ mij}$$

$$(-5\bar{v}_1 + 3\bar{v}_2 + 4\bar{v}_3 = \bar{0}) \text{ } \left. \begin{array}{l} \text{1x, hvorfor?} \\ * \end{array} \right)$$

Bemerk

Når $\bar{0}$ er med i et vektorsystem, er der lineær afhængighed.

Vektorsystem med to vektorer ($\neq \bar{0}$)

\bar{v}_1 og \bar{v}_2 lin. afh. $\Leftrightarrow \exists r : \bar{v}_2 = r\bar{v}_1$

$\Leftrightarrow \bar{v}_1$ og \bar{v}_2 er proportionale

*

$$\dots \sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 4 & 12 & -4 \\ 0 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 4 & 0 & 5 \\ 0 & 4 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{sat } x_3 = 4r ; \quad (x_1, x_2, x_3) = r(-5, 3, 4), r \in \mathbb{R}$$

Ekvivalente udsagn om $A \in \mathbb{R}^{m \times n}$ ②

a) A's sifler er lin. uafh.

b) $\forall \bar{b} \in \mathbb{R}^m$: $A\bar{x} = \bar{b}$ har ligst én lsm.

c) nullitet $A = 0$

d) rang $A = n$

e) alle sifler i den reducedech echelonform er basisvektorer i \mathbb{R}^m

f) $A\bar{x} = \bar{0}$ har kun nullsm.

g) alle sifler i A er pivot-sifler

bewis for f \Rightarrow g:

antag \bar{u} og \bar{v} begge er lsm. til $A\bar{x} = \bar{b}$

$$\Rightarrow A(\bar{u} - \bar{v}) = \bar{0} \Rightarrow \bar{u} - \bar{v} = 0 \Rightarrow \bar{u} = \bar{v}$$

$\Rightarrow A\bar{x} = \bar{b}$ har ligst én lsm.

Homogen ligningsystem $A\bar{x} = \bar{0}$

Har altid nullsm. (den triviale lsm.)

$\Rightarrow A\bar{x} = \bar{0}$ har én eller uendelig mange lsm'er

$$\text{ehr } 3x_1 + x_2 - 5x_3 = 0$$

$$x_2 + 4x_3 = 0$$

$$\begin{bmatrix} 3 & 1 & -5 \\ 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & -9 \\ 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 4 \end{bmatrix}$$

$$(x_1, x_2, x_3) = v(3, -4, 1), v \in \mathbb{R}$$

Løsningsmængde: $\text{Sp}\{(3, -4, 1)\}$

sætn. om linear afhængighed

$\bar{v}_1, \bar{v}_2, \dots, \bar{v}_k$ lin. afh. $\Leftrightarrow \exists \bar{v}_j$ (mindst ét) :

$$\bar{v}_j \in S_p \{ \bar{v}_1, \dots, \bar{v}_{j-1}, \bar{v}_{j+1}, \dots, \bar{v}_k \}$$

Endvidere: $\bar{v}_i \neq \bar{0} \Rightarrow \exists j > 1 : \bar{v}_j \in S_p \{ \bar{v}_1, \dots, \bar{v}_{j-1} \}$

beweis

$\Rightarrow : \bar{v}_1, \dots, \bar{v}_k$ lin. afh.

i $\bar{v}_i = \bar{0}$: sætningen er triviale opfyldt, da

$$\bar{v}_1 = 0\bar{v}_1 + \dots + 0\bar{v}_k$$

ii $\bar{v}_i \neq \bar{0}$:

lin. afh. $\Rightarrow \exists (r_1, \dots, r_k) \neq (0, \dots, 0)$:

$$r_1\bar{v}_1 + \dots + r_k\bar{v}_k = \bar{0}$$

valg største j , så $r_j \neq 0$, dvs.

$$r_j \neq 0 \wedge r_{j+1} = \dots = r_k = 0$$

venner $j > 1$, da $j=1 \Rightarrow r_1\bar{v}_1 = \bar{0} \Rightarrow \bar{v}_1 = \bar{0}$

modstrid

$$j > 1 \wedge r_j \neq 0 \wedge r_{j+1} = \dots = r_k = 0 \Rightarrow$$

$$\bar{v}_j = -\frac{r_1}{r_j}\bar{v}_1 - \dots - \frac{r_{j-1}}{r_j}\bar{v}_{j-1}$$

$$\Leftarrow : \bar{v}_j = s_1\bar{v}_1 + \dots + s_{j-1}\bar{v}_{j-1} + s_{j+1}\bar{v}_{j+1} + \dots + s_k\bar{v}_k$$

$$\Leftarrow s_1\bar{v}_1 + \dots + s_{j-1}\bar{v}_{j-1} - \bar{v}_j + s_{j+1}\bar{v}_{j+1} + \dots + s_k\bar{v}_k = \bar{0}$$

↑

koefficienten til \bar{v}_j er $-1 \neq 0$

dvs. $\bar{v}_1, \dots, \bar{v}_k$ lin. afh.

satte:

$$\bar{v}_1, \dots, \bar{v}_k \in \mathbb{R}^m$$

$n > m \Rightarrow \bar{v}_1, \dots, \bar{v}_n$ lin. afh.

bevis

$\text{rang } [\bar{v}_1 \dots \bar{v}_k] \leq m < k \Rightarrow \bar{v}_1, \dots, \bar{v}_k$ er lin. afh.

eks. $\bar{v}_1 = (1, 0, 1), \bar{v}_2 = (-1, 1, 2), \bar{v}_3 = (2, 1, 3), \bar{v}_4 = (1, -2, 4)$
er lin. afh. (fire vektorer fra \mathbb{R}^3)

Oversigt $A \in \mathbb{R}^{m \times n}$

	# løsninger til $A\bar{x} = \bar{v}$	A' 's rigtige vektorer	den reducerede række-eckelonform
$\text{rang } A = m$	$\forall \bar{v}: \text{mindsten lsm.}$	udsparer \mathbb{R}^m	alle rækker har pivotel.
$\text{rang } A = n$	$\forall \bar{v}: \text{majsten lsm.}$	lin. uafh.	alle rækker har pivotel.

skitser

$\text{rang } A = m$

$$\left[\begin{array}{cccc|c} x & & & & \\ & x & & & \\ & & x & & \\ & & & x & \\ & & & & x \end{array} \right]$$

$\text{rang } A = n$

$$\left[\begin{array}{ccccc|c} x & & & & & \\ & x & & & & \\ & & x & & & \\ & & & x & & \\ & & & & x & \end{array} \right]$$