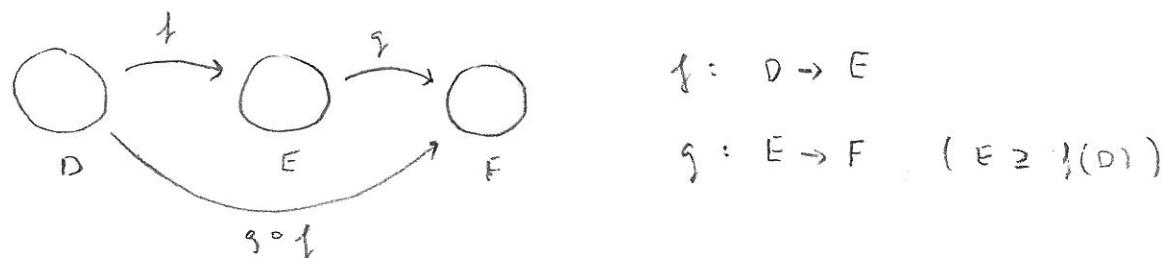


## Sammensat afbillede



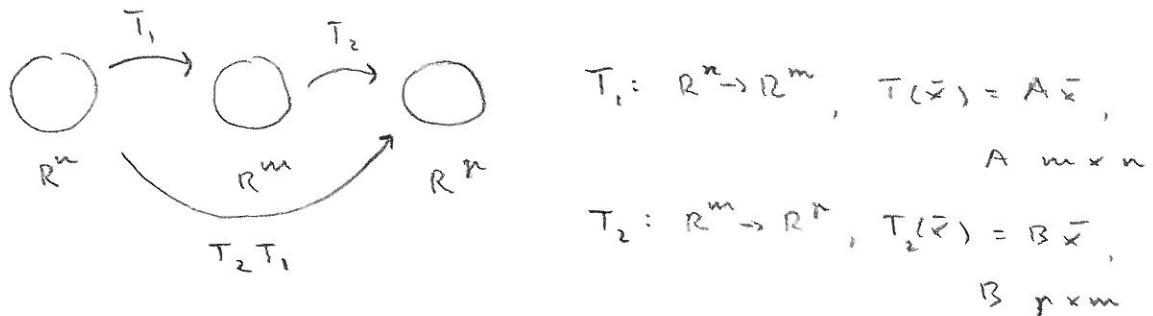
$$g \circ f: D \rightarrow F, \quad g \circ f(x) = g(f(x))$$

ekse.  $f: R \rightarrow R, \quad f(x) = 3x$

$g: R \rightarrow R, \quad g(x) = \sin x$

$g \circ f: R \rightarrow R, \quad g \circ f(x) = \sin 3x$

## Sammenset lineær transformation



venmark notation  $T_2 \circ T_1 := T_2 T_1$

$$T_2 T_1(\bar{x}) = T_2(A\bar{x}) = B(A\bar{x}) = BA\bar{x}, \quad \text{der}$$

$$T_2 T_1: R^m \rightarrow R^n, \quad T_2 T_1(\bar{x}) = BA\bar{x}, \quad BA = n \times n$$

ekse.  $T_1: R_2 \rightarrow R_2, \quad T_1(\bar{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \bar{x}$

$$T_2: R_2 \rightarrow R_2, \quad T_2(\bar{x}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \bar{x}$$

$$T_2 T_1: R_2 \rightarrow R_2, \quad T_2 T_1(\bar{x}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \bar{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \bar{x}$$

$$T_1 T_2: R_2 \rightarrow R_2, \quad T_1 T_2(\bar{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \bar{x} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \bar{x}$$

Bemerk dog, at når både  $T_2 T_1$  og  $T_1 T_2$  er defineret, så gælder i alm., at  $T_2 T_1 \neq T_1 T_2$ .

els.  $T_1: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $T_1(\bar{x}) = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \bar{x}$

$$T_2: \mathbb{R}^2 \rightarrow \mathbb{R}^2, T_2(\bar{x}) = \begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix} \bar{x}$$

$$T_2 T_1: \mathbb{R}^2 \rightarrow \mathbb{R}^2, T_2 T_1(\bar{x}) = \begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \bar{x} = \begin{bmatrix} -1 & 5 \\ 15 & -5 \end{bmatrix} \bar{x}$$

$$T_1 T_2: \mathbb{R}^2 \rightarrow \mathbb{R}^2, T_1 T_2(\bar{x}) = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix} \bar{x} = \begin{bmatrix} 2 & 9 \\ 6 & -8 \end{bmatrix} \bar{x}$$

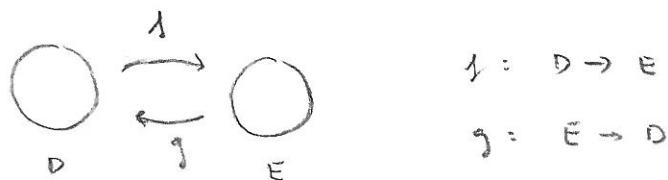
els.  $T_1: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ ,  $T_1(\bar{x}) = A \bar{x}$ ,  $A \in \mathbb{R}^{3 \times 2}$

$$T_2: \mathbb{R}^3 \rightarrow \mathbb{R}^4, T_2(\bar{x}) = B \bar{x}, B \in \mathbb{R}^{4 \times 3}$$

$$T_2 T_1: \mathbb{R}^2 \rightarrow \mathbb{R}^4, T_2 T_1(\bar{x}) = BA \bar{x}, BA \in \mathbb{R}^{4 \times 2}$$

$T_1 T_2$  är inte definierat

## Inversa afbildning



$g$  är inversa afb. till  $f$ , där

$$\forall x \in D: g \circ f(x) = x$$

$$\forall y \in E: f \circ g(y) = y$$

notation:  $g := f^{-1}$

els.

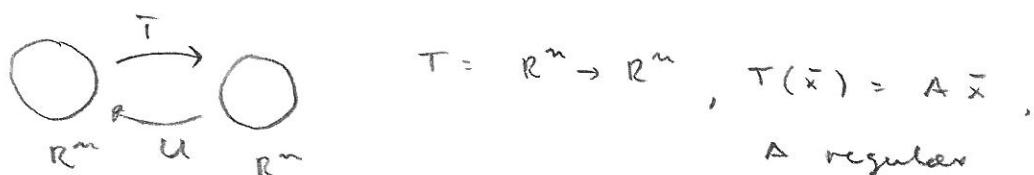
$$f: \mathbb{R} \rightarrow \mathbb{R}_+, f(x) = \exp x$$

$$g: \mathbb{R}_+ \rightarrow \mathbb{R}, g(y) = \ln y$$

$$\forall x \in \mathbb{R}: \ln \circ \exp(x) = x$$

$$\forall y \in \mathbb{R}_+: \exp \circ \ln(y) = y$$

## Invers linear operator



Sat  $U: \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $U(\bar{x}) = A^{-1} \bar{x}$ , se remark

$$\forall \bar{x} \in \mathbb{R}^n: U T(\bar{x}) = A^{-1} A(\bar{x}) = I \bar{x} = \bar{x}$$

$$\forall \bar{x} \in \mathbb{R}^n: T U(\bar{x}) = A A^{-1}(\bar{x}) = I \bar{x} = \bar{x}$$

} dvs.

$$U = T^{-1}$$

$$\text{ehn. } T: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad T(\bar{x}) = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \bar{x}$$

$$T^{-1}: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad T^{-1}(x) = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}^{-1} \bar{x} = \frac{1}{-5} \begin{bmatrix} 1 & -3 \\ -2 & 1 \end{bmatrix} \bar{x}$$

$$= \frac{1}{5} \begin{bmatrix} -1 & 3 \\ 2 & -1 \end{bmatrix} \bar{x}$$

Fleme etsempler

$$\text{ehn. } T_1: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad T_1(\bar{x}) = \begin{bmatrix} 1 & 1 \\ 1 & -3 \\ 4 & 0 \end{bmatrix} \bar{x}$$

$$T_2: \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad T_2(\bar{x}) = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 3 & 0 \end{bmatrix} \bar{x}$$

$$T_2 T_1: \mathbb{R}^3 \rightarrow \mathbb{R}^2, \quad T_2 T_1(\bar{x}) = \begin{bmatrix} 1 & -1 & 4 \\ 1 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -3 \\ 4 & 0 \end{bmatrix} \bar{x} = \begin{bmatrix} 16 & 4 \\ 4 & -8 \end{bmatrix} \bar{x}$$

$$T_1 T_2: \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad T_1 T_2(\bar{x}) = \begin{bmatrix} 1 & 1 \\ 1 & -3 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 1 & 3 & 0 \end{bmatrix} \bar{x} = \begin{bmatrix} 2 & 2 & 4 \\ -2 & 10 & 4 \\ 4 & -4 & 16 \end{bmatrix} \bar{x}$$

$$\text{ehn. } T: \mathbb{R}^3 \rightarrow \mathbb{R}^3: \quad T(\bar{x}) = \begin{bmatrix} 4 & 1 & -1 \\ -1 & -1 & 0 \\ -5 & -3 & 1 \end{bmatrix} \bar{x}$$

$$\left[ \begin{array}{ccc|ccc} 4 & 1 & -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ -5 & -3 & 1 & 0 & 0 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -3 & -1 & 1 & 4 & 0 \\ 0 & 2 & 1 & 0 & -5 & 1 \end{array} \right]$$

$$\sim \left[ \begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & -1 & 1 \\ 0 & 2 & 1 & 0 & -5 & 1 \end{array} \right] \sim \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & 2 & -7 & 3 \end{array} \right]$$

$$T^{-1}: \mathbb{R}^3 \rightarrow \mathbb{R}^3: \quad T^{-1}(\bar{x}) = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \\ 2 & -7 & 3 \end{bmatrix} \bar{x}$$

$$\text{kontrol: } \begin{bmatrix} 1 & -2 & 1 \\ -1 & 1 & -1 \\ 2 & -7 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 & -1 \\ -1 & -1 & 0 \\ -5 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## Determinanter

Determinant af  $2 \times 2$  matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Determinant af  $n \times n$  matrix  $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$   
 $\det A ?$

Først nogle definitioner

Undermatrix til  $A$ :

$$A_{ij} = \left[ \begin{array}{|c|c|c|c|} \hline & & \dots & \\ \hline & \cancel{a_{ij}} & & \\ \hline & & & \\ \hline \end{array} \right] \quad \text{er } A \text{ med } i^{\text{te}} \text{ række  
og } j^{\text{te}} \text{ spalte slettet, der.  
} A_{ij} \text{ er } (n-1) \times (n-1)$$

Underdeterminant:  $\det A_{ij}$

Komplementet  $c_{ij}$  hørende til elementet  $a_{ij}$

$$c_{ij} = (-1)^{i+j} \det A_{ij}, \quad \text{bemerk, at } (-1)^{i+j} \text{ følger et skæbroadtumuster}$$

$$\begin{bmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ & & & & + \end{bmatrix}$$

Rekursiv definition af  $\det A$

$$\det A = \sum_{j=1}^n a_{ij} c_{ij}, \quad i \in \{1, \dots, n\}$$

$$= \sum_{i=1}^n a_{ij} c_{ij}, \quad j \in \{1, \dots, m\}$$

(oplossning efter en række eller en spalte)

exs

$$\begin{vmatrix} 2 & -1 & 3 \\ 1 & 4 & -2 \\ -1 & 0 & 1 \end{vmatrix} = (-1) \begin{vmatrix} -1 & 3 \\ 4 & -2 \end{vmatrix} - 0 + 1 \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix}$$

$$= -(2 - 12) + (8 + 1) = 10 + 9 = 19$$

exs

$$\begin{vmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{vmatrix} = -0 + (-1) \begin{vmatrix} 1 & -2 & 5 \\ 3 & 1 & 0 \\ 0 & 4 & -2 \end{vmatrix} - 2 \begin{vmatrix} 1 & -2 & 5 \\ 2 & 0 & 4 \\ 0 & 4 & -2 \end{vmatrix} + 0$$

$$= -15 \begin{vmatrix} 3 & 1 \\ 0 & 4 \end{vmatrix} - 0 + (-2) \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} - 7(1 \begin{vmatrix} 0 & 4 \\ 4 & -2 \end{vmatrix} - 2 \begin{vmatrix} -2 & 5 \\ 4 & -2 \end{vmatrix})$$

$$= -(5 \cdot 12 - 2 \cdot 2) - 7(1(-16) - 2(-16))$$

$$= -40 - 112 = -152$$

Determinant of singular blockmatrix

$\begin{vmatrix} A & B \\ 0 & I \end{vmatrix}$ , ved successiv optorning efter sidste  
række fra

$$\begin{vmatrix} A & B \\ 0 & I \end{vmatrix} = \det A$$

Determinant af triangulære matricer

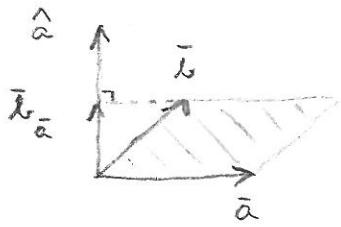
$$\begin{vmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ 0 & 0 & u_{33} & \dots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_{nn} \end{vmatrix} = u_{11} u_{22} u_{33} \dots u_{nn} = \prod_{i=1}^n u_{ii}$$

(successiv optorning efter 1. sijle)

$$\begin{vmatrix} l_{11} & 0 & 0 & \dots & 0 \\ l_{21} & l_{22} & 0 & \dots & 0 \\ l_{31} & l_{32} & l_{33} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \dots & l_{nn} \end{vmatrix} = l_{11} l_{22} l_{33} \dots l_{nn} = \prod_{i=1}^n l_{ii}$$

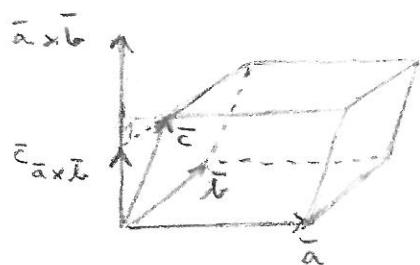
(successiv optorning efter 1. række)

## Areal of parallelogram



$$\begin{aligned}
 A_r &= h g = \left| \bar{b} \cdot \frac{\hat{a}}{\|\hat{a}\|} \right| \|\bar{a}\| \\
 &= |\hat{a} \cdot \bar{b}| = |(-a_2, a_1) \cdot (b_1, b_2)| \\
 &= |a_1 b_2 - a_2 b_1| = \left\| \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \right\| \\
 &= |\det A|, \text{ where } A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}
 \end{aligned}$$

## Volumen of parallelepipedum



$$\begin{aligned}
 V &= h G = \left| \bar{c} \cdot \frac{\bar{a} \times \bar{b}}{\|\bar{a} \times \bar{b}\|} \right| \|\bar{a} \times \bar{b}\| \\
 &= |\bar{a} \times \bar{b} \cdot \bar{c}|
 \end{aligned}$$

$$\begin{aligned}
 \bar{a} \times \bar{b} &= \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad (\text{symbolisch schreiben}) \\
 &= \left( \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right)
 \end{aligned}$$

$$\begin{aligned}
 V &= \left| \left( \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right) \cdot (c_1, c_2, c_3) \right| \\
 &= \left| c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right|
 \end{aligned}$$

$$= \left| \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \right| = |\det A|, \text{ where}$$

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

Ombytning af to rækker i A

Først to naborækker  $A \rightarrow B$

$$\begin{bmatrix} a_{ij} \\ a_{i+1,j} \end{bmatrix} \rightarrow \begin{bmatrix} a_{i+1,j} \\ a_{ij} \end{bmatrix} \begin{array}{l} i^{\text{te}} \text{ række} \\ (i+1)^{\text{te}} \text{ række} \end{array}$$

det B udregnes ved oplossning efter  $(i+1)^{\text{te}}$  række

$$\begin{aligned} \det B &= \sum_j a_{ij} (-1)^{i+1+j} \det A_{ij} = - \sum_j a_{ij} (-1)^{i+1} \det A_{ij} \\ &= - \det A \end{aligned}$$

Dernæst to vilk. rækker  $A \rightarrow B$

$$\begin{bmatrix} a_{ij} \\ e_{ij} \end{bmatrix} \begin{array}{l} \# \text{ nabo ombytninger :} \\ k + (k-1) = 2k-1, \text{ der altid ujægtig} \\ \Rightarrow \det B = - \det A \end{array}$$

To ens rækker i A

Ombytning af de to ens rækker :  $\det A = -\det A$   
 $\Rightarrow \det A = 0$

Mult. af række i A med konstant  $A \rightarrow B$

$$\begin{bmatrix} a_{ij} \end{bmatrix} \rightarrow \begin{bmatrix} ka_{ij} \end{bmatrix}$$

$$\det B = \sum_i (ka_{ij}) c_{ii} = k \sum_i a_{ij} c_{ii} = k \det A$$

Mult. af A med konstant  $A \rightarrow B$

$$\det B = k^n \det A, \text{ da } n \text{ rækker}$$

Addition af et multiplum af en række i A til en anden række  $A \rightarrow B$

$$\begin{bmatrix} a_{ij} \\ a_{ij} \end{bmatrix} \rightarrow \begin{bmatrix} a_{ij} \\ a_{ij} + k a_{ij} \end{bmatrix}$$

$$\begin{aligned} \det B &= \sum_j (a_{ij} + k a_{ij}) c_{ij} \\ &= \sum_j a_{ij} c_{ij} + k \sum_j a_{ij} c_{ij} \\ &= \det A + k \cdot 0 \\ &= \det A \end{aligned}$$

eks.

$$\begin{vmatrix} 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -2 \\ -2 & 0 & 4 & -7 \\ 4 & -4 & 4 & 15 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 3 & -3 \\ 0 & 0 & 4 & -2 \\ -2 & 0 & 4 & -7 \\ 0 & -4 & 12 & 1 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1 & 3 & -3 \\ 0 & 4 & -2 \\ -4 & 12 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 3 & -3 \\ 0 & 4 & -2 \\ 0 & 24 & -11 \end{vmatrix}$$

$$= -2 \cdot 1 \begin{vmatrix} 4 & -2 \\ 24 & -11 \end{vmatrix} = -2 (-44 + 48) = -8$$

eks.

$$\begin{vmatrix} 0 & -1 & 2 & 2 \\ 1 & -1 & 0 & -2 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 0 & -1 & 2 & 2 \\ 1 & -1 & 0 & -2 \\ 0 & 3 & 0 & 5 \\ 0 & 0 & 2 & -3 \end{vmatrix}$$

$$= -1 \begin{vmatrix} -1 & 2 & 2 \\ 3 & 0 & 5 \\ 0 & 2 & -3 \end{vmatrix} = - \begin{vmatrix} -1 & 2 & 2 \\ 0 & 6 & 11 \\ 0 & 2 & -3 \end{vmatrix}$$

$$= -(-1) \cdot 2 \cdot 20 = -40$$