1.1 a $\quad \operatorname{Cov}(A\underset{\sim}{x}, B\underset{\sim}{y}) = E[(A\underset{\sim}{x} - A\underset{\sim}{\mu_x})(B\underset{\sim}{y} - B\underset{\sim}{\mu_y})^T]$

$$= E[A(\underset{\sim}{x} - \underset{\sim}{\mu_x})(\underset{\sim}{y} - \underset{\sim}{\mu_y})^T B^T]$$

$$= A\, E[(\underset{\sim}{x} - \underset{\sim}{\mu_x})(\underset{\sim}{y} - \underset{\sim}{\mu_y})^T]\, B^T$$

$$= A \operatorname{Cov}(\underset{\sim}{x}, \underset{\sim}{y})\, B^T$$

b $\quad \operatorname{Cov}(\underset{\sim}{x}, \underset{\sim}{y}) = E[(\underset{\sim}{x} - \underset{\sim}{\mu_x})(\underset{\sim}{y} - \underset{\sim}{\mu_y})^T]$

$$= E[\underset{\sim}{x}\underset{\sim}{y}^T - \underset{\sim}{x}\underset{\sim}{\mu_y}^T - \underset{\sim}{\mu_x}\underset{\sim}{y}^T + \underset{\sim}{\mu_x}\underset{\sim}{\mu_y}^T]$$

$$= E[\underset{\sim}{x}\underset{\sim}{y}^T] - \underset{\sim}{\mu_x}\underset{\sim}{\mu_y}^T - \underset{\sim}{\mu_x}\underset{\sim}{\mu_y}^T + \underset{\sim}{\mu_x}\underset{\sim}{\mu_y}^T$$

$$= E[\underset{\sim}{x}\underset{\sim}{y}^T] - E\underset{\sim}{x}\, E\underset{\sim}{y}^T$$

c $\quad \operatorname{Var}(\underset{\sim}{x} - \underset{\sim}{a}) = E[(\underset{\sim}{x} - \underset{\sim}{a} - (\underset{\sim}{\mu_x} - \underset{\sim}{a}))(\underset{\sim}{x} - \underset{\sim}{a} - (\underset{\sim}{\mu_x} - \underset{\sim}{a}))^T]$

$$= E[(\underset{\sim}{x} - \underset{\sim}{\mu_x})(\underset{\sim}{x} - \underset{\sim}{\mu_x})^T]$$

$$= \operatorname{Var} \underset{\sim}{x}$$

1.2 $\quad \sum_i (\underset{\sim}{y_i} - \bar{\underset{\sim}{y}})(\underset{\sim}{y_i} - \bar{\underset{\sim}{y}})^T = \sum_i (\underset{\sim}{y_i}\underset{\sim}{y_i}^T - \underset{\sim}{y_i}\bar{\underset{\sim}{y}}^T - \bar{\underset{\sim}{y}}\underset{\sim}{y_i}^T + \bar{\underset{\sim}{y}}\bar{\underset{\sim}{y}}^T)$

$$= \sum_i \underset{\sim}{y_i}\underset{\sim}{y_i}^T - n\bar{\underset{\sim}{y}}\bar{\underset{\sim}{y}}^T - n\bar{\underset{\sim}{y}}\bar{\underset{\sim}{y}}^T + n\bar{\underset{\sim}{y}}\bar{\underset{\sim}{y}}^T$$

$$= \sum_i \underset{\sim}{y_i}\underset{\sim}{y_i}^T - n\bar{\underset{\sim}{y}}\bar{\underset{\sim}{y}}^T$$

1.3 a $\quad \operatorname{Var}(\sum_i a_i \underset{\sim}{x_i}) = \operatorname{Cov}(\sum_i a_i \underset{\sim}{x_i}, \sum_j a_j \underset{\sim}{x_j})$

$$= \sum_i \sum_j a_i a_j \operatorname{Cov}(\underset{\sim}{x_i}, \underset{\sim}{x_j})$$

$$= \sum_i a_i^2 \operatorname{Var} \underset{\sim}{x_i} + \sum_i \sum_{j \neq i} a_i a_j\, 0$$

$$= \left(\sum_i a_i^2\right) \Sigma$$

b $\quad \operatorname{Cov}(\sum_i a_i \underset{\sim}{x_i}, \sum_j b_j \underset{\sim}{x_j}) = \sum_i \sum_j a_i b_j \operatorname{Cov}(\underset{\sim}{x_i}, \underset{\sim}{x_j})$

$$= \sum_i a_i b_i \operatorname{Var} \underset{\sim}{x_i} + \sum_i \sum_{j \neq i} a_i b_j\, 0$$

$$= \left(\sum_i a_i b_i\right) \Sigma$$

$$= 0$$

$$\Leftrightarrow \sum_i a_i b_i = 0 \quad , \quad \text{as} \quad \Sigma > 0$$

1.4 $\quad E\underset{\sim}{x} = \underset{\sim}{\theta}\ , \quad \operatorname{Var} \underset{\sim}{x} = \Sigma$

$$E[\underset{\sim}{x}^T A \underset{\sim}{x}] = E\left[\sum_i \sum_j a_{ij} x_i x_j\right] = \sum_i \sum_j a_{ij} E[x_i x_j]$$

$$= \sum_i \sum_j a_{ij} \left(\operatorname{cov}(x_i, x_j) + E x_i E x_j \right)$$

$$= \sum_i \sum_j a_{ij} \sigma_{ij} + \sum_i \sum_j a_{ij} \theta_i \theta_j$$

$$= \sum_i \sum_j a_{ij} \sigma_{ji} + \underset{\sim}{\theta}^T A \underset{\sim}{\theta}$$

$$= \sum_i (A\Sigma)_{ii} + \underset{\sim}{\theta}^T A \underset{\sim}{\theta}$$

$$= \operatorname{tr}(A\Sigma) + \underset{\sim}{\theta}^T A \underset{\sim}{\theta}$$

1.5 $\quad \underset{\sim}{y} = K\underset{\sim}{\beta} + \underset{\sim}{u} \quad K \ n \times p \quad \operatorname{rank} K = p \quad E\underset{\sim}{u} = \underset{\sim}{0} \quad P = K(K^TK)^{-1}K^T$

a $\quad P^T = (K^T)^T (K^T (K^T)^T)^{-1} K^T = K(K^TK)^{-1}K^T = P$

$$P^2 = K(K^TK)^{-1}K^T K (K^TK)^{-1}K^T = K I_p (K^TK)^{-1}K^T$$

$$= K(K^TK)^{-1}K = P$$

$$(I_n - P)^T = I_n^T - P^T = I_n - P$$

$$(I_n - P)^2 = (I_n - P)(I_n - P) = I_n - P - P + P = I_n - P$$

b $\quad (I_n - P)K = K - PK = K - K(K^TK)^{-1}K^TK$

$$= K - KI_p = K - K = 0$$

c $\quad \operatorname{rank}(I_n - P) = \operatorname{tr}(I_n - P) = n - \operatorname{tr} P = n - \operatorname{rank} P$

$$= n - p \qquad (\text{cf. A6.3})$$

d $\quad \underset{\sim}{y}^T (I_n - P) \underset{\sim}{y} = (K\underset{\sim}{\beta} + \underset{\sim}{u})^T (I_n - P)(K\underset{\sim}{\beta} + \underset{\sim}{u})$

$$= \underset{\sim}{\beta}^T K^T (I_n - P) K \underset{\sim}{\beta} + \underset{\sim}{\beta}^T K^T (I_n - P) \underset{\sim}{u}$$

$$+ \underset{\sim}{u}^T (I_n - P) K \underset{\sim}{\beta} + \underset{\sim}{u}^T (I_n - P) \underset{\sim}{u}$$

$$= \underset{\sim}{\beta}^T K^T O \underset{\sim}{\beta} + \underset{\sim}{\beta}^T O^T \underset{\sim}{u}$$

$$+ \underset{\sim}{u}^T O \underset{\sim}{\beta} + \underset{\sim}{u}^T (I_n - P) \underset{\sim}{u}$$

$$= 0 + 0 + 0 + \underset{\sim}{u}^T (I_n - P) \underset{\sim}{u}$$

$$= \underset{\sim}{u}^T (I_n - P) \underset{\sim}{u}$$

Extra (Ch. 1)

$$Y = \sum_i \sum_j a_{ij} \underline{x}_i \underset{\sim}{x}_j^T$$

Rename $a_{ij} := \frac{1}{2}(a_{ij} + a_{ji})$

$$y_{rs} = \sum_i \sum_j a_{ij} x_{ir} x_{js} = \sum_i x_{ir} \sum_j a_{ij} x_{js} = \sum_i x_{ir} (AX)_{is}$$

$$= \sum_i x^T{}_{ri} (AX)_{is} = (X^T A X)_{rs}$$

Hence $Y = X^T A X$, A symmetric

1.6 $\underset{\sim}{x}_1, \dots, \underset{\sim}{x}_n$ sample from d-dim. distribution

$$g_{rs} = (\underset{\sim}{x}_r - \bar{\underset{\sim}{x}})^T \hat{\Sigma}^{-1} (\underset{\sim}{x}_s - \bar{\underset{\sim}{x}}) , \quad \hat{\Sigma} = \frac{1}{n} Q = \frac{1}{n} \tilde{X}^T \tilde{X}$$

a
$$\begin{aligned}
\sum_r g_{rr} &= \sum_r (\underset{\sim}{x}_r - \bar{\underset{\sim}{x}})^T \hat{\Sigma}^{-1} (\underset{\sim}{x}_r - \bar{\underset{\sim}{x}}) \\
&= \sum_r \operatorname{tr} (\underset{\sim}{x}_r - \bar{\underset{\sim}{x}})^T \hat{\Sigma}^{-1} (\underset{\sim}{x}_r - \bar{\underset{\sim}{x}}) \\
&= \sum_r \operatorname{tr} \hat{\Sigma}^{-1} (\underset{\sim}{x}_r - \bar{\underset{\sim}{x}}) (\underset{\sim}{x}_r - \bar{\underset{\sim}{x}})^T \\
&= \operatorname{tr} \left(\hat{\Sigma}^{-1} \sum_r (\underset{\sim}{x}_r - \bar{\underset{\sim}{x}}) (\underset{\sim}{x}_r - \bar{\underset{\sim}{x}})^T \right) \\
&= \operatorname{tr} (\hat{\Sigma}^{-1} n \hat{\Sigma}) \\
&= n \operatorname{tr} (I_d) \\
&= nd
\end{aligned}$$

$$E[g_{rr}] = \frac{1}{n} E\left[\sum_r g_{rr} \right] = \frac{1}{n} nd = d$$

b
$$\begin{aligned}
\sum_s g_{rs} &= (\underset{\sim}{x}_r - \bar{\underset{\sim}{x}})^T \hat{\Sigma}^{-1} \sum_s (\underset{\sim}{x}_s - \bar{\underset{\sim}{x}}) \\
&= (\bar{\underset{\sim}{x}}_r - \bar{\underset{\sim}{x}})^T \hat{\Sigma}^{-1} \underset{\sim}{0} \\
&= 0
\end{aligned}$$

$$\sum_s g_{rs} = g_{rr} + \sum_{s \neq r} g_{rs} \iff \sum_{s \neq r} g_{rs} = \sum_s g_{rs} - g_{rr}$$

$$\implies \sum_{s \neq r} g_{rs} = 0 - g_{rr}$$

$$E[g_{rs}] = \frac{1}{n-1} E\left[\sum_{r \neq s} g_{rs} \right] = \frac{1}{n-1} E[-g_{rr}]$$

$$= -\frac{d}{n-1}$$