3.8

When $H_{02}: \Sigma = \sigma^2 I_d$ is true we have, cf. p. 61 l. 2 with $\Sigma = \sigma^2 I_d$,

$$\ln L(\hat{\mu}, \sigma^2) = -\frac{nd}{2}\ln(2\pi) - \frac{1}{2}\left(n \ln(\det(\sigma^2 I_d)) + \mathrm{tr}\left(\frac{1}{\sigma^2} I_d Q\right)\right)$$

$$= -\frac{nd}{2}\ln(2\pi) - \frac{1}{2}\left(n \ln((\sigma^2)^d) + \mathrm{tr}\left(\frac{1}{\sigma^2} Q\right)\right)$$

$$= -\frac{nd}{2}\ln(2\pi) - \frac{1}{2}\left(nd \ln \sigma^2 + \frac{1}{\sigma^2}\mathrm{tr}\, Q\right)$$

$$\frac{d \ln L}{d\sigma^2} = -\frac{1}{2}\left(\frac{nd}{\sigma^2} - \frac{1}{\sigma^4}\mathrm{tr}\, Q\right) = 0 \quad \text{for } \sigma^2 = \hat{\sigma}^2 = \frac{\mathrm{tr}\, Q}{nd} \qquad *$$

$$\ln L(\hat{\mu}, \hat{\sigma}^2) = -\frac{nd}{2}\ln(2\pi) - \frac{1}{2}\left(nd \ln \frac{\mathrm{tr}\, Q}{nd} + nd\right)$$

$$= -\frac{nd}{2}\left(\ln(2\pi) + \ln\frac{\mathrm{tr}\, Q}{nd} + 1\right)$$

$$L(\hat{\mu}, \hat{\sigma}^2) = \left(2\pi e \frac{\mathrm{tr}\, Q}{nd}\right)^{-\frac{nd}{2}}$$

$$\ell_2 = \frac{L(\hat{\mu}, \hat{\sigma}^2)}{L(\hat{\mu}, \hat{\Sigma})} = \frac{\left(2\pi e \frac{\mathrm{tr}\, Q}{nd}\right)^{-\frac{nd}{2}}}{(2\pi e)^{-\frac{nd}{2}}\left(\det \frac{Q}{n}\right)^{-\frac{n}{2}}} = \frac{\left(\frac{\mathrm{tr}\, Q}{d}\right)^{-\frac{nd}{2}}}{(\det Q)^{-\frac{n}{2}}}$$

$$\Lambda_2 = \ell_2^{\frac{2}{n}} = \frac{\det Q}{\left(\frac{\mathrm{tr}\, Q}{d}\right)^d}$$

3.9

When $H_{03}: \Sigma = I_d$ is true we have, cf. p. 61 l. 2 with $\Sigma = I_d$,

$$\ln L(\hat{\mu}) = -\frac{nd}{2}\ln(2\pi) - \frac{1}{2} n \ln(\det I_d) - \frac{1}{2}\mathrm{tr}(I_d Q)$$

$$= -\frac{nd}{2}\ln(2\pi) - \frac{1}{2}\mathrm{tr}\, Q$$

$$L(\hat{\mu}) = (2\pi)^{-\frac{nd}{2}} e^{-\frac{1}{2}\mathrm{tr}\, Q}$$

$$\ell_3 = \frac{L(\hat{\mu})}{L(\hat{\mu}, \hat{\Sigma})} = \frac{(2\pi)^{-\frac{nd}{2}} e^{-\frac{1}{2}\mathrm{tr}\, Q}}{(2\pi e)^{-\frac{nd}{2}}\left(\det \frac{Q}{n}\right)^{-\frac{n}{2}}}$$

$$= e^{\frac{nd}{2}}\left(\frac{1}{n}\right)^{\frac{nd}{2}} (\det Q)^{\frac{n}{2}} e^{-\frac{1}{2}\mathrm{tr}\, Q}$$

$$= \left(\frac{e}{n}\right)^{\frac{nd}{2}} (\det Q)^{\frac{n}{2}} \exp\left(-\frac{1}{2}\mathrm{tr}\, Q\right)$$

3.10

When $H_{04}: \underset{\sim}{\mu} = \underset{\sim}{\mu}_0,\ \Sigma = \Sigma_0$ is true we have, cf. formula (3.2) with (3.3) and (3.4) p. 60,

$$L = (2\pi)^{-\frac{nd}{2}}(\det \Sigma_0)^{-\frac{n}{2}} \exp\left(-\frac{1}{2}\mathrm{tr}\left(\Sigma_0^{-1}\left(Q + n(\underset{\sim}{\bar{x}} - \underset{\sim}{\mu}_0)(\underset{\sim}{\bar{x}} - \underset{\sim}{\mu}_0)^T\right)\right)\right)$$

$$\ell_4 = \frac{L}{L(\hat{\mu}, \hat{\Sigma})} = \frac{(2\pi)^{-\frac{nd}{2}}(\det \Sigma_0)^{-\frac{n}{2}} \exp\left(-\frac{1}{2}\mathrm{tr}\left(\Sigma_0^{-1} Q + n\Sigma_0^{-1}(\underset{\sim}{\bar{x}} - \underset{\sim}{\mu}_0)(\underset{\sim}{\bar{x}} - \underset{\sim}{\mu}_0)^T\right)\right)}{(2\pi e)^{-\frac{nd}{2}}\left(\det \frac{Q}{n}\right)^{-\frac{n}{2}}}$$

$$= \left(\frac{e}{n}\right)^{\frac{nd}{2}} \left(\det(\Sigma_0^{-1} Q)\right)^{\frac{n}{2}} \exp\left(-\frac{1}{2}\left(\mathrm{tr}\,\Sigma_0^{-1} Q + n(\underset{\sim}{\bar{x}} - \underset{\sim}{\mu}_0)^T \Sigma_0^{-1}(\underset{\sim}{\bar{x}} - \underset{\sim}{\mu}_0)\right)\right)$$

* $\frac{d^2 \ln L}{d(\sigma^2)^2} = \frac{1}{2}\frac{nd}{\sigma^4} - \frac{\mathrm{tr}\, Q}{\sigma^6}$; $\frac{d^2 \ln L}{d(\sigma^2)^2}\left(\frac{\mathrm{tr}\, Q}{nd}\right) = \frac{1}{2}\frac{(nd)^3}{(\mathrm{tr}\, Q)^2} - \frac{(nd)^3\,\mathrm{tr}\, Q}{(\mathrm{tr}\, Q)^3} = -\frac{1}{2}\frac{(nd)^3}{(\mathrm{tr}\, Q)^2} < 0 \Rightarrow$ max.

3.11

$$\underset{\sim}{x}_i = \begin{bmatrix} \underset{\sim}{x}_{i1} \\ \underset{\sim}{x}_{i2} \\ \vdots \\ \underset{\sim}{x}_{ib} \end{bmatrix} \qquad \underset{\sim}{x}_{ij} \quad d_0\text{-dim.} \quad , \quad j = 1, \dots, b$$

$$b\, d_0 = d$$

$$\bar{\underset{\sim}{x}}_{i\cdot} = \frac{1}{b} \sum_{j=1}^{b} \underset{\sim}{x}_{ij} \qquad d_0\text{-dim.}$$

Let $$\underset{\sim}{y}_i = \begin{bmatrix} \bar{\underset{\sim}{x}}_{i\cdot} \\ \underset{\sim}{x}_{i2} - \underset{\sim}{x}_{i\cdot} \\ \vdots \\ \underset{\sim}{x}_{ib} - \underset{\sim}{x}_{i\cdot} \end{bmatrix} = C\, \underset{\sim}{x}_i \quad , \qquad \underset{\sim}{y}_i \quad d\text{-dim.}$$

$$C \quad d \times d$$

$$C = \frac{1}{b} \begin{bmatrix} I_{d_0} & I_{d_0} & \cdots & I_{d_0} \\ -I_{d_0} & (b-1)\, I_{d_0} & \cdots & -I_{d_0} \\ \vdots & & & \vdots \\ -I_{d_0} & -I_{d_0} & \cdots & (b-1)\, I_{d_0} \end{bmatrix}$$

$$\sim \frac{1}{b} \begin{bmatrix} I_{d_0} & I_{d_0} & \cdots & I_{d_0} \\ 0 & b\, I_{d_0} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b\, I_{d_0} \end{bmatrix} \qquad \text{(by row operations)}$$

$$\det C = \left(\frac{1}{b}\right)^{d} b^{(b-1)\, d_0} = b^{-d_0} \neq 0$$

$\Rightarrow$ C non-singular