3.12 $\underset{\sim}{x}_i \sim N_d(\underset{\sim}{\mu}, \Sigma)$, $i = 1, \dots, n$, independent

$$\Sigma = \sigma^2 \begin{bmatrix} 1 & \rho & \dots & \rho \\ \rho & 1 & \dots & \rho \\ \vdots & \vdots & & \vdots \\ \rho & \rho & \dots & 1 \end{bmatrix} = \sigma^2 \Sigma_1$$

We look for a linear transformation $\underset{\sim}{y}_i = T^T \underset{\sim}{x}_i$, where T diagonalizes Σ.

$$\Sigma_1 - \lambda I_d = \begin{bmatrix} 1-\lambda & \rho & \dots & \rho \\ \rho & 1-\lambda & \dots & \rho \\ \vdots & \vdots & & \vdots \\ \rho & \rho & \dots & 1-\lambda \end{bmatrix}$$

$$\det(\Sigma_1 - \lambda I_d) = (1-\lambda-\rho)^{d-1}(1-\lambda+(d-1)\rho) \quad \text{jf A 3.5}$$
$$= (-1)^d (\lambda - (1-\rho))^{d-1} (\lambda - (1+(d-1)\rho))$$

Eigenvalues for Σ_1 : $\lambda = \begin{cases} 1+(d-1)\rho \\ 1-\rho \quad \text{(multiplicity } d-1) \end{cases}$

By usual methods we can find an orthogonal matrix which has eigenvectors corresponding to the eigenvalues as columns.

$$T = \begin{bmatrix} \frac{1}{\sqrt{d}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \dots & \frac{1}{\sqrt{(d-1)(d-2)}} & \frac{1}{\sqrt{d(d-1)}} \\ \frac{1}{\sqrt{d}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \dots & \frac{1}{\sqrt{(d-1)(d-2)}} & \frac{1}{\sqrt{d(d-1)}} \\ \frac{1}{\sqrt{d}} & 0 & -\sqrt{\frac{2}{3}} & \dots & \frac{1}{\sqrt{(d-1)(d-2)}} & \frac{1}{\sqrt{d(d-1)}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{\sqrt{d}} & 0 & 0 & \dots & -\sqrt{\frac{d-2}{d-1}} & \frac{1}{\sqrt{d(d-1)}} \\ \frac{1}{\sqrt{d}} & 0 & 0 & \dots & 0 & -\sqrt{\frac{d-1}{d}} \end{bmatrix}$$

3.12

$$\underset{\sim}{y}_i = T^T \underset{\sim}{x}_i \sim N_d(T^T \underset{\sim}{\mu}, \Lambda), \quad \Lambda = T^T \Sigma T$$
$$= \sigma^2 T^T \Sigma_1 T$$

$$\Lambda = \sigma^2 \begin{bmatrix} 1+(d-1)\rho & 0 & \dots & 0 \\ 0 & 1-\rho & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1-\rho \end{bmatrix}$$

$$= \begin{bmatrix} a+(d-1)b & 0 & \dots & 0 \\ 0 & a-b & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & a-b \end{bmatrix}, \quad \begin{aligned} a &= \sigma^2 \\ b &= \sigma^2\rho \end{aligned}$$

$$\ln L_Y(T^T\hat{\mu}, a, b) = -\frac{nd}{2}\ln(2\pi) - \frac{n}{2}\ln\det\Lambda - \frac{1}{2}\operatorname{tr}(\Lambda^{-1}Q_Y),$$

cf. page 61 line 2

$$\det\Lambda = (a+(d-1)b)(a-b)^{d-1}$$

$$\Lambda^{-1}Q_Y = \begin{bmatrix} \frac{1}{a+(d-1)b} & 0 & \dots & 0 \\ 0 & \frac{1}{a-b} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \frac{1}{a-b} \end{bmatrix} Q_Y$$

$$\operatorname{tr}(\Lambda^{-1}Q_Y) = \frac{q_1}{a+(d-1)b} + \frac{q_2}{a-b}, \quad \begin{aligned} q_1 &= q_{11}^Y \\ q_2 &= \sum_{j=2}^{d} q_{jj}^Y \end{aligned}$$

$$\ln L_Y(T^T\hat{\mu}, a, b) = -\frac{nd}{2}\ln(2\pi)$$
$$- \frac{n}{2}\left(\ln(a+(d-1)b) + (d-1)\ln(a-b)\right)$$
$$- \frac{1}{2}\left(\frac{q_1}{a+(d-1)b} + \frac{q_2}{a-b}\right)$$

$$-2\,\frac{\partial \ln L_Y}{\partial a} = \frac{n}{a+(d-1)b} + \frac{n(d-1)}{a-b} - \frac{q_1}{(a+(d-1)b)^2} - \frac{q_2}{(a-b)^2} = 0$$

$$-2\,\frac{\partial \ln L_Y}{\partial b} = \frac{n(d-1)}{a+(d-1)b} - \frac{n(d-1)}{a-b} - \frac{(d-1)q_1}{(a+(d-1)b)^2} + \frac{q_2}{(a-b)^2} = 0$$

Addition of the equations leads to

$$\frac{nd}{a+(d-1)b} - \frac{dq_1}{(a+(d-1)b)^2} = 0 \quad \Rightarrow \quad q_1 = n(a+(d-1)b)$$

Inserting in the first equation leads to

$$\frac{n(d-1)}{a-b} - \frac{q_2}{(a-b)^2} = 0 \quad \Rightarrow \quad q_2 = n(d-1)(a-b)$$
$$= n\left((d-1)a - (d-1)b\right)$$

These two new equations are solved wrt a and b

$$q_1 + q_2 = nda \quad \Rightarrow \quad a = \frac{1}{nd}(q_1+q_2) = \frac{1}{nd}\operatorname{tr} Q_Y$$

$$q_1 - \frac{1}{d-1}q_2 = ndb \quad \Rightarrow \quad b = \frac{1}{nd}\left(q_1 - \frac{1}{d-1}q_2\right)$$
$$= \frac{1}{nd}\left(\frac{d}{d-1}q_1 - \frac{1}{d-1}\operatorname{tr} Q_Y\right)$$

$$\hat{a} = \frac{1}{nd}\operatorname{tr} Q = \frac{1}{nd}\sum_j q_{jj} = \frac{n-1}{n}\,\frac{1}{d}\sum_j s_{jj}$$

$$\hat{b} = \frac{1}{nd(d-1)}\left(d\,\frac{1}{d}\,\underset{\sim}{1}_d^T Q\, \underset{\sim}{1}_d - \operatorname{tr} Q\right)$$
$$= \frac{1}{nd(d-1)}\left(\sum_j\sum_k q_{jk} - \sum_j q_{jj}\right) = \frac{1}{nd(d-1)}\sum_j\sum_{k\neq j} q_{jk}$$
$$= \frac{n-1}{n}\,\frac{1}{d(d-1)}\sum_j\sum_{k\neq j} s_{jk}$$

$$\hat{\sigma}^2 = \frac{n-1}{n}\,\frac{1}{d}\sum_j s_{jj}$$

$$\widehat{\sigma^2\rho} = \frac{n-1}{n}\,\frac{1}{d(d-1)}\sum_j\sum_{k\neq j} s_{jk}$$

$$\hat{\rho} = \frac{\sum_j\sum_{k\neq j} s_{jk}}{(d-1)\sum_j s_{jj}}$$

Alternative method :

$$\Sigma = \begin{bmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & & \vdots \\ b & b & \cdots & a \end{bmatrix} , \qquad \begin{aligned} a &= \sigma^2 \\ b &= \sigma^2 \rho \end{aligned}$$

$$= (a-b) I_d + b \, \underset{\sim}{1}_d \underset{\sim}{1}_d^T$$

$$\ln L(\hat{\mu}, a, b) = -\frac{nd}{2} \ln(2\pi) - \frac{n}{2} \ln \det \Sigma - \frac{1}{2} \operatorname{tr}(\Sigma^{-1} Q) ,$$

cf. page 61 line 2

$$\det \Sigma = (a-b)^{d-1} (a + (d-1)b) \qquad \text{cf. A 3 5}$$

$$\Sigma^{-1} = \frac{1}{a-b} I_d + \frac{-b}{(a-b)(a+(d-1)b)} \underset{\sim}{1}_d \underset{\sim}{1}_d^T \qquad \text{cf. A 3.5}$$

$$\Sigma^{-1} Q = \frac{1}{a-b} Q - \frac{b}{(a-b)(a+(d-1)b)} \underset{\sim}{1}_d \underset{\sim}{1}_d^T Q$$

$$\operatorname{tr}(\Sigma^{-1} Q) = \frac{1}{a-b} \operatorname{tr} Q - \frac{b}{(a-b)(a+(d-1)b)} \underset{\sim}{1}_d^T Q \underset{\sim}{1}_d$$

$$= \frac{(a+(d-2)b) q_0 - bq}{(a-b)(a+(d-1)b)} , \qquad \begin{aligned} q_0 &= \operatorname{tr} Q = \sum_j q_{jj} \\ q &= \underset{\sim}{1}_d^T Q \underset{\sim}{1}_d - \operatorname{tr} Q \\ &= \sum_j \sum_{k \neq j} q_{jk} \end{aligned}$$

$$\begin{aligned} \ln L(\hat{\mu}, a, b) = & -\frac{nd}{2} \ln(2\pi) \\ & - \frac{n}{2} \left((d-1) \ln (a-b) + \ln (a + (d-1)b) \right) \\ & - \frac{1}{2} \frac{(a+(d-2)b) q_0 - bq}{(a-b)(a+(d-1)b)} \end{aligned}$$

By use of computer algebra for differentiation of ln L w r t a and b and for solution of the equation system $\frac{\partial \ln L}{\partial a} = 0$, $\frac{\partial \ln L}{\partial b} = 0$

$$\hat{a} = \frac{1}{nd} q_0 = \frac{1}{nd} \sum_j q_{jj} = \frac{n-1}{n} \frac{1}{d} \sum_j s_{jj}$$

$$\hat{b} = \frac{1}{nd(d-1)} q = \frac{1}{nd(d-1)} \sum_j \sum_{k \neq j} q_{jk} = \frac{n-1}{n} \frac{1}{d(d-1)} \sum_j \sum_{k \neq j} s_{jk}$$

$$\hat{\sigma}^2 = \frac{n-1}{n} \frac{1}{d} \sum_j s_{jj}$$

$$\widehat{\sigma^2 \rho} = \frac{n-1}{n} \frac{1}{d(d-1)} \sum_j \sum_{k \neq j} s_{jk}$$

$$\hat{\rho} = \frac{\sum_j \sum_{k \neq j} s_{jk}}{(d-1) \sum_j s_{jj}}$$

3.13 Notation etc. as in exercise 3.12

Note that $L(\hat{\mu}, a, b) = L_Y(T^T\hat{\mu}, a, b)$, as

- $\det \Sigma = \det \Lambda$

- $$\operatorname{tr}(\Sigma^{-1}Q) = \operatorname{tr}(TT^T\Sigma^{-1}TT^TQ) = \operatorname{tr}((T^T\Sigma T)^{-1}T^TQT) = \operatorname{tr}(\Lambda^{-1}Q_Y)$$

$$\operatorname{tr}\left((\Lambda(\hat{a},\hat{b}))^{-1}Q_Y\right) = \frac{q_1(\hat{a},\hat{b})}{\hat{a}+(d-1)\hat{b}} + \frac{q_2(\hat{a},\hat{b})}{\hat{a}-\hat{b}} = n + n(d-1) = nd$$

$$L_Y(T^T\hat{\mu}, \hat{a}, \hat{b}) = (2\pi)^{-\frac{nd}{2}}\left((\hat{a}+(d-1)\hat{b})(\hat{a}-\hat{b})^{d-1}\right)^{-\frac{n}{2}} e^{-\frac{nd}{2}}$$
$$= (2\pi e)^{-\frac{nd}{2}}\left((\hat{\sigma}^2+(d-1)\hat{\sigma}^2\hat{\rho})(\hat{\sigma}^2-\hat{\sigma}^2\hat{\rho})^{d-1}\right)^{-\frac{n}{2}}$$
$$= (2\pi e)^{-\frac{nd}{2}}\left((\hat{\sigma}^2)^d(1-\hat{\rho})^{d-1}(1+(d-1)\hat{\rho})\right)^{-\frac{n}{2}}$$

$$\ell_6 = \frac{L(\hat{\mu},\hat{a},\hat{b})}{L(\hat{\mu},\hat{\Sigma})} = \frac{L_Y(T^T\hat{\mu},\hat{a},\hat{b})}{L(\hat{\mu},\hat{\Sigma})}$$
$$= \frac{(2\pi e)^{-\frac{nd}{2}}\left((\hat{\sigma}^2)^d(1-\hat{\rho})^{d-1}(1+(d-1)\hat{\rho})\right)^{-\frac{n}{2}}}{(2\pi e)^{-\frac{nd}{2}}\left(\det\frac{Q}{n}\right)^{-\frac{n}{2}}}$$

$$\ell_6^{\frac{2}{n}} = \frac{\det\left(\frac{n-1}{n}S\right)}{(\hat{\sigma}^2)^d(1-\hat{\rho})^{d-1}(1+(d-1)\hat{\rho})} = \frac{\left(\frac{n-1}{n}\right)^d \det S}{(\hat{\sigma}^2)^d(1-\hat{\rho})^{d-1}(1+(d-1)\hat{\rho})}$$

$$= \frac{\left(\frac{n-1}{n}\right)^d \det S}{\left(\frac{n-1}{nd}\operatorname{tr} S\right)^d\left(1-\frac{\underset{\sim}{1}_d^T S \underset{\sim}{1}_d - \operatorname{tr} S}{(d-1)\operatorname{tr} S}\right)^{d-1}\left(1+(d-1)\frac{\underset{\sim}{1}_d^T S \underset{\sim}{1}_d - \operatorname{tr} S}{(d-1)\operatorname{tr} S}\right)}$$

$$= \frac{d^d \operatorname{tr} S}{\left(\operatorname{tr} S - \frac{\underset{\sim}{1}_d^T S \underset{\sim}{1}_d - \operatorname{tr} S}{d-1}\right)^{d-1}\left(\operatorname{tr} S + \underset{\sim}{1}_d S \underset{\sim}{1}_d - \operatorname{tr} S\right)}$$

$$= \frac{(d-1)^{d-1} d^d \operatorname{tr} S}{\left(d\operatorname{tr} S - \underset{\sim}{1}_d^T S \underset{\sim}{1}_d\right)^{d-1}\underset{\sim}{1}_d^T S \underset{\sim}{1}_d}$$

Alternatively (cf. alternative solution of exercise 3.12)

$$\operatorname{tr}\left((\Sigma(\hat{a},\hat{b}))^{-1}Q\right) = \frac{(\hat{a}+(d-2)\hat{b})\,q_0(\hat{a}) - \hat{b}\,q(\hat{b})}{(\hat{a}-\hat{b})(\hat{a}+(d-1)\hat{b})}$$

$$= \frac{(\hat{a}+(d-2)\hat{b})\,md\,\hat{a} - \hat{b}\,md\,(d-1)\,\hat{b}}{(\hat{a}-\hat{b})(\hat{a}+(d-1)\hat{b})}$$

$$= \frac{md\,(\hat{a}^2+(d-2)\hat{a}\hat{b}+(d-1)\hat{b}^2)}{(\hat{a}-\hat{b})(\hat{a}+(d-1)\hat{b})}$$

$$= md$$

Other calculations no change