3.15

$$\left.\begin{array}{ll} \underset{\sim}{v}_i \sim N_d(\underset{\sim}{\mu}_1, \Sigma), & i = 1, \dots, n_1 \\ \underset{\sim}{w}_j \sim N_d(\underset{\sim}{\mu}_2, \Sigma), & j = 1, \dots, n_2 \end{array}\right\} \text{ all independent}$$

$$L(\hat{\underset{\sim}{\mu}}_1, \hat{\underset{\sim}{\mu}}_2, \hat{\Sigma}) = (2\pi e)^{-\frac{nd}{2}} (\det \hat{\Sigma})^{-\frac{n}{2}}, \quad \hat{\Sigma} = \frac{Q}{n} = \frac{Q_1 + Q_2}{n_1 + n_2},$$

cf. page 104 line 9

Under H_0: $\underset{\sim}{\mu}_1 = \underset{\sim}{\mu}_2 \; (= \underset{\sim}{\mu})$

$$\hat{\underset{\sim}{\mu}} = \frac{n_1 \bar{\underset{\sim}{v}} + n_2 \bar{\underset{\sim}{w}}}{n}$$ cf. page 60 line 2 from bottom

$$\hat{\Sigma}_0 = \frac{Q_0}{n}$$ cf. formula (3.5) page 61

$$= \frac{1}{n} \Big(\sum_{i=1}^{n_1} \Big(\underset{\sim}{v}_i - \frac{n_1 \bar{\underset{\sim}{v}} + n_2 \bar{\underset{\sim}{w}}}{n}\Big)\Big(\underset{\sim}{v}_i - \frac{n_1 \bar{\underset{\sim}{v}} + n_2 \bar{\underset{\sim}{w}}}{n}\Big)^T + \sum_{j=1}^{n_2} \Big(\underset{\sim}{w}_j - \frac{n_1 \bar{\underset{\sim}{v}} + n_2 \bar{\underset{\sim}{w}}}{n}\Big)\Big(\underset{\sim}{w}_j - \frac{n_1 \bar{\underset{\sim}{v}} + n_2 \bar{\underset{\sim}{w}}}{n}\Big)^T \Big)$$

$$= \frac{1}{n} \Big(\sum_{i=1}^{n_1} \Big(\underset{\sim}{v}_i - \bar{\underset{\sim}{v}} + \bar{\underset{\sim}{v}} - \frac{n_1 \bar{\underset{\sim}{v}} + n_2 \bar{\underset{\sim}{w}}}{n}\Big)\Big(\underset{\sim}{v}_i - \bar{\underset{\sim}{v}} + \bar{\underset{\sim}{v}} - \frac{n_1 \bar{\underset{\sim}{v}} + n_2 \bar{\underset{\sim}{w}}}{n}\Big)^T + \sum_{j=1}^{n_2} \Big(\underset{\sim}{w}_j - \bar{\underset{\sim}{w}} + \bar{\underset{\sim}{w}} - \frac{n_1 \bar{\underset{\sim}{v}} + n_2 \bar{\underset{\sim}{w}}}{n}\Big)\Big(\underset{\sim}{w}_j - \bar{\underset{\sim}{w}} + \bar{\underset{\sim}{w}} - \frac{n_1 \bar{\underset{\sim}{v}} + n_2 \bar{\underset{\sim}{w}}}{n}\Big)^T \Big)$$

$$= \frac{1}{n} \Big(\sum_{i=1}^{n_1} \Big(\underset{\sim}{v}_i - \bar{\underset{\sim}{v}} + \frac{n_2(\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})}{n}\Big)\Big(\underset{\sim}{v}_i - \bar{\underset{\sim}{v}} + \frac{n_2(\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})}{n}\Big)^T + \sum_{j=1}^{n_2} \Big(\underset{\sim}{w}_j - \bar{\underset{\sim}{w}} + \frac{n_1(\bar{\underset{\sim}{w}} - \bar{\underset{\sim}{v}})}{n}\Big)\Big(\underset{\sim}{w}_j - \bar{\underset{\sim}{w}} + \frac{n_1(\bar{\underset{\sim}{w}} - \bar{\underset{\sim}{v}})}{n}\Big)^T \Big)$$

$$= \frac{1}{n} \Big(Q_1 + 0 + 0 + \frac{n_1 n_2^2 (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})(\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})^T}{n^2} + Q_2 + 0 + 0 + \frac{n_2 n_1^2 (\bar{\underset{\sim}{w}} - \bar{\underset{\sim}{v}})(\bar{\underset{\sim}{w}} - \bar{\underset{\sim}{v}})^T}{n^2} \Big)$$

$$= \frac{1}{n} \Big(Q_1 + Q_2 + \frac{n_1 n_2 (n_2 + n_1)}{n^2} (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})(\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})^T \Big)$$

$$= \frac{1}{n} \Big(Q + \frac{n_1 n_2}{n} (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})(\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})^T \Big)$$

$$L(\hat{\underset{\sim}{\mu}}, \hat{\Sigma}_0) = (2\pi e)^{-\frac{nd}{2}} (\det \hat{\Sigma}_0)^{-\frac{n}{2}}$$ cf. page 61 formula (3.6)

$$\ell = \frac{L(\hat{\underset{\sim}{\mu}}, \hat{\Sigma}_0)}{L(\hat{\underset{\sim}{\mu}}, \hat{\Sigma})} = \frac{(2\pi e)^{-\frac{nd}{2}} (\det \hat{\Sigma}_0)^{-\frac{n}{2}}}{(2\pi e)^{-\frac{nd}{2}} (\det \hat{\Sigma})^{-\frac{n}{2}}} = \left(\frac{\det \hat{\Sigma}}{\det \hat{\Sigma}_0} \right)^{\frac{n}{2}}$$

$$\ell^{\frac{2}{n}} = \frac{\det \hat{\Sigma}}{\det \hat{\Sigma}_0} = \frac{\det\left(\frac{Q}{n}\right)}{\det\left(\frac{1}{n}\left(Q + \frac{n_1 n_2}{n} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})(\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})^T\right)\right)}$$

$$= \frac{n^{-d} \det I_d}{n^{-d} \det\left(I_d + \frac{n_1 n_2}{n} Q^{-1} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})(\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})^T\right)}$$

$$= \frac{1}{\det\left(I_1 + \frac{n_1 n_2}{n} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})^T Q^{-1} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})\right)}$$

cf. hint to ex. 2.12 a (problems 4)

$$= \frac{1}{1 + \frac{n_1 n_2}{n} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})^T Q^{-1} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})}$$

$$= \frac{1}{1 + \frac{n_1 n_2}{(n_1+n_2-2)n} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})^T S_p^{-1} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})}, \quad S_p = \frac{Q}{n_1+n_2-2}$$

$$= \frac{1}{1 + \frac{1}{n_1+n_2-2} T^2}$$ with

$$T^2 = \frac{n_1 n_2}{n} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}})^T S_p^{-1} (\underset{\sim}{\bar{v}} - \underset{\sim}{\bar{w}}) \sim T^2(d, n_1+n_2-2)$$

cf. corollary to th. 2.8

ℓ is a decreasing function of T^2

Alternative calculation of $\hat{\mu}$ og $\hat{\Sigma}_0$:

$$\ln L(\underset{\sim}{\mu}, \Sigma) = -\frac{nd}{2}\ln(2\pi) - \frac{n}{2}\ln\det\Sigma - \frac{1}{2}\operatorname{tr}(\Sigma^{-1}Q)$$
$$-\frac{1}{2}\operatorname{tr}\left(\Sigma^{-1}\left(n_1(\underset{\sim}{\bar{v}}-\underset{\sim}{\mu})(\underset{\sim}{\bar{v}}-\underset{\sim}{\mu})^T + n_2(\underset{\sim}{\bar{w}}-\underset{\sim}{\mu})(\underset{\sim}{\bar{w}}-\underset{\sim}{\mu})^T\right)\right)$$

cf. page 103 formula (3.76) with $\underset{\sim}{\mu_1} = \underset{\sim}{\mu_2}$

$$\frac{d}{d\underset{\sim}{\mu}}\operatorname{tr}\left(\Sigma^{-1}\left(n_1(\underset{\sim}{\bar{v}}-\underset{\sim}{\mu})(\underset{\sim}{\bar{v}}-\underset{\sim}{\mu})^T + n_2(\underset{\sim}{\bar{w}}-\underset{\sim}{\mu})(\underset{\sim}{\bar{w}}-\underset{\sim}{\mu})^T\right)\right)$$

$= -2\Sigma^{-1}\left(n_1(\underset{\sim}{\bar{v}}-\underset{\sim}{\mu}) + n_2(\underset{\sim}{\bar{w}}-\underset{\sim}{\mu})\right)$ cf. A 1.1 (6) and A 8.1 (6)

$$= \underset{\sim}{0} \quad \text{for} \quad \underset{\sim}{\mu} = \underset{\sim}{\hat{\mu}} = \frac{n_1\underset{\sim}{\bar{v}} + n_2\underset{\sim}{\bar{w}}}{n}$$

$$n_1(\underset{\sim}{\bar{v}}-\underset{\sim}{\hat{\mu}})(\underset{\sim}{\bar{v}}-\underset{\sim}{\hat{\mu}})^T + n_2(\underset{\sim}{\bar{w}}-\underset{\sim}{\hat{\mu}})(\underset{\sim}{\bar{w}}-\underset{\sim}{\hat{\mu}})^T$$
$$= n_1\frac{n_2(\underset{\sim}{\bar{v}}-\underset{\sim}{\bar{w}})}{n}\frac{n_2(\underset{\sim}{\bar{v}}-\underset{\sim}{\bar{w}})^T}{n} + n_2\frac{n_1(\underset{\sim}{\bar{w}}-\underset{\sim}{\bar{v}})}{n}\frac{n_1(\underset{\sim}{\bar{w}}-\underset{\sim}{\bar{v}})^T}{n}$$
$$= \frac{n_1n_2}{n}\frac{n_2+n_1}{n}(\underset{\sim}{\bar{v}}-\underset{\sim}{\bar{w}})(\underset{\sim}{\bar{v}}-\underset{\sim}{\bar{w}})^T = \frac{n_1n_2}{n}(\underset{\sim}{\bar{v}}-\underset{\sim}{\bar{w}})(\underset{\sim}{\bar{v}}-\underset{\sim}{\bar{w}})^T$$

$$\ln L(\hat{\mu}, \Sigma) = -\frac{nd}{2}\ln(2\pi) - \frac{n}{2}\ln\det\Sigma$$
$$-\frac{1}{2}\operatorname{tr}\left(\Sigma^{-1}\left(Q + \frac{n_1n_2}{n}(\underset{\sim}{\bar{v}}-\underset{\sim}{\bar{w}})(\underset{\sim}{\bar{v}}-\underset{\sim}{\bar{w}})^T\right)\right)$$

$\hat{\Sigma}_0 = \frac{1}{n}\left(Q + \frac{n_1n_2}{n}(\underset{\sim}{\bar{v}}-\underset{\sim}{\bar{w}})(\underset{\sim}{\bar{v}}-\underset{\sim}{\bar{w}})^T\right)$ cf. page 61 formula (3.5)

3.17

$$\left.\begin{array}{ll} \underset{\sim}{v}_i \sim N_d(\underset{\sim}{\mu}_1, \Sigma_1) , & i = 1, \dots, n_1 \\ \underset{\sim}{w}_j \sim N_d(\underset{\sim}{\mu}_2, \Sigma_2) , & j = 1, \dots, n_2 \end{array}\right\} \text{ all independent}$$

$$n_1 < n_2$$

$$\Sigma_1 = k\,\Sigma_2 \quad , \quad k \text{ is known}$$

$$H_0 : \underset{\sim}{\mu}_1 = \underset{\sim}{\mu}_2 \quad , \quad H_1 : \underset{\sim}{\mu}_1 \neq \underset{\sim}{\mu}_2$$

$$\bar{\underset{\sim}{v}} \sim N_d\left(\underset{\sim}{\mu}_1, \frac{\Sigma_1}{n_1}\right)$$

$$\bar{\underset{\sim}{w}} \sim N_d\left(\underset{\sim}{\mu}_2, \frac{\Sigma_2}{n_2}\right) = N_d\left(\underset{\sim}{\mu}_2, \frac{\Sigma_1}{k n_2}\right)$$

$$Q_1 \sim W_d(n_1 - 1, \Sigma_1)$$

$$Q_2 \sim W_d(n_2 - 1, \Sigma_2) \Rightarrow k Q_2 \sim W_d(n_2 - 1, \Sigma_1)$$

$$\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}} \sim N_d\left(\underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2, \left(\frac{1}{n_1} + \frac{1}{k n_2}\right)\Sigma_1\right)$$

$$Q_1 + k Q_2 \sim W_d(n_1 + n_2 - 2, \Sigma_1)$$

$$T^2 = \frac{n_1 + n_2 - 2}{\frac{1}{n_1} + \frac{1}{k n_2}} (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})^T (Q_1 + k Q_2)^{-1} (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})$$

$$= \frac{k n_1 n_2 (n_1 + n_2 - 2)}{n_1 + k n_2} (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})^T (Q_1 + k Q_2)^{-1} (\bar{\underset{\sim}{v}} - \bar{\underset{\sim}{w}})$$

$$\sim T^2(d, n_1 + n_2 - 2) \qquad \text{cf. corollary to theorem 2.8}$$

3.18

$$\underset{\sim}{x}_i \sim N_d(\underset{\sim}{\mu}, \Sigma), \quad i = 1, \dots, n, \quad \text{independent}$$

$$H_0: \Sigma = \Sigma_0$$

$$H_{0\ell}: \underset{\sim}{\ell}^T \Sigma \underset{\sim}{\ell} = \underset{\sim}{\ell}^T \Sigma_0 \underset{\sim}{\ell}, \quad \underset{\sim}{\ell} \neq \underset{\sim}{0}$$

$\bigcap_{\ell} H_{0\ell}$ is equivalent with H_0

$$\forall \underset{\sim}{\ell} \neq \underset{\sim}{0}: \quad \frac{\underset{\sim}{\ell}^T Q \underset{\sim}{\ell}}{\underset{\sim}{\ell}^T \Sigma_0 \underset{\sim}{\ell}} \sim \chi^2(n-1) \quad \text{when } H_0 \text{ is true},$$

cf. corollary to theorem 2.2

Acceptance area for H_0:

$$\bigcap_{\ell} \left\{ X \;\middle|\; \chi^2_{\frac{\alpha}{2}}(n-1) \leq \frac{\underset{\sim}{\ell}^T Q \underset{\sim}{\ell}}{\underset{\sim}{\ell}^T \Sigma_0 \underset{\sim}{\ell}} \leq \chi^2_{1-\frac{\alpha}{2}}(n-1) \right\}$$

$$= \left\{ X \;\middle|\; \chi^2_{\frac{\alpha}{2}}(n-1) \leq \inf_{\underset{\sim}{\ell}} \frac{\underset{\sim}{\ell}^T Q \underset{\sim}{\ell}}{\underset{\sim}{\ell}^T \Sigma_0 \underset{\sim}{\ell}} \leq \sup_{\underset{\sim}{\ell}} \frac{\underset{\sim}{\ell}^T Q \underset{\sim}{\ell}}{\underset{\sim}{\ell}^T \Sigma_0 \underset{\sim}{\ell}} \leq \chi^2_{1-\frac{\alpha}{2}}(n-1) \right\}$$

$$= \left\{ X \;\middle|\; \chi^2_{\frac{\alpha}{2}}(n-1) \leq \varphi_n \leq \varphi_1 \leq \chi^2_{1-\frac{\alpha}{2}}(n-1) \right\},$$

where φ_1 is the largest and φ_1 is the smallest eigenvalue corresponding to $\Sigma_0^{-1} Q$, cf. A 7.5