8.5 $Y = XB + U$, $\underset{\sim}{u}_i \sim N_d(\underset{\sim}{0}, \Sigma)$, $i = 1, \ldots, n$, indep.

X $n \times p$, rank $X = r$

$H_0: AB = C$, A $q \times p$, rank $A = q$

$E = \tilde{Y}(I_n - P_\Omega)\tilde{Y} \sim W_d(n-r, \Sigma)$, $\tilde{Y} = Y - XB_0$ *)

$H = \tilde{Y}(P_\Omega - P_\omega)\tilde{Y} \sim W_d(q, \Sigma; \Sigma^{-\frac{1}{2}} D \Sigma^{-\frac{1}{2}})$, where

$D = (AB - C)^T (A(X^TX)^{-1}A^T)^{-1}(AB - C) \geq 0$

(equality only when H_0 is true)

E and H are independent, cf modified corollary 2 to theorem 2.4

$$E[E^{-1}] = \frac{\Sigma^{-1}}{(n-r)-d-1} \quad \text{cf. ex. 2.23 (i)}$$

$E[H] = q\Sigma + D$ cf. formula (8.49)

$$E[E^{-1}H] = E[E^{-1}]\,E[H], \quad \text{E and H independent} \Rightarrow E^{-1} \text{ and H independent}$$

$$= \frac{\Sigma^{-1}}{n-r-d-1}(q\Sigma + D)$$

$$= \frac{1}{n-r-d-1}(q I_d + \Sigma^{-1}D)$$

$$\Sigma^{-1}D = (n-r-d-1)\,E[E^{-1}H] - q I_d$$

Hence

$$\widehat{\Sigma^{-1}D} = (n-r-d-1)\,E^{-1}H - q I_d$$

is an unbiased estimate for $\Sigma^{-1}D$

*) B_0 is any fixed solution to $AB = C$

8.7

$$Y = XB + U, \quad \underset{\sim}{u}_i \sim N_d(\underset{\sim}{0}, \Sigma), \; i = 1, \dots, n, \; \text{indep.}$$

$$XB = \underset{\sim}{1}_n \underset{\sim}{\beta}_1^T + X_2 B_2 = [\underset{\sim}{1}_n \; X_2] \begin{bmatrix} \underset{\sim}{\beta}_1^T \\ B_2 \end{bmatrix}, \qquad X \; n \times p, \; \text{rank } X = p$$

$$X_2^T \underset{\sim}{1}_n = \underset{\sim}{0}$$

a

$$\hat{B} = (X^T X)^{-1} X^T Y = \left(\begin{bmatrix} \underset{\sim}{1}_n^T \\ X_2^T \end{bmatrix} [\underset{\sim}{1}_n \; X_2] \right)^{-1} \begin{bmatrix} \underset{\sim}{1}_n^T \\ X_2^T \end{bmatrix} Y$$

$$= \begin{bmatrix} n & \underset{\sim}{0}^T \\ \underset{\sim}{0} & X_2^T X_2 \end{bmatrix}^{-1} \begin{bmatrix} \underset{\sim}{1}_n^T Y \\ X_2^T Y \end{bmatrix} = \begin{bmatrix} \frac{1}{n} & \underset{\sim}{0}^T \\ \underset{\sim}{0} & (X_2^T X_2)^{-1} \end{bmatrix} \begin{bmatrix} n \bar{\underset{\sim}{y}}^T \\ X_2^T Y \end{bmatrix}$$

$$= \begin{bmatrix} \bar{\underset{\sim}{y}}^T \\ (X_2^T X_2)^{-1} X_2^T Y \end{bmatrix}$$

$$\Leftrightarrow \quad \hat{\underset{\sim}{\beta}}_1 = \bar{\underset{\sim}{y}} \quad \wedge \quad \hat{B}_2 = (X_2^T X_2)^{-1} X_2^T Y$$

$$\hat{B}_2 = (X_2^T X_2)^{-1} (X_2^T - \tfrac{1}{n} X_2^T \underset{\sim}{1}_n \underset{\sim}{1}_n^T) Y, \quad \text{as } X_2^T \underset{\sim}{1}_n = \underset{\sim}{0}$$

$$= (X_2^T X_2)^{-1} X_2^T (I_n - \tfrac{1}{n} \underset{\sim}{1}_n \underset{\sim}{1}_n^T) Y$$

$$= (X_2^T X_2)^{-1} X_2^T \tilde{Y}, \quad \text{where}$$

$$\tilde{Y} = (I_n - \tfrac{1}{n} \underset{\sim}{1}_n \underset{\sim}{1}_n^T) Y = Y - \tfrac{1}{n} \underset{\sim}{1}_n \underset{\sim}{1}_n^T Y = Y - \underset{\sim}{1}_n \bar{\underset{\sim}{y}}^T$$

$$= [\underset{\sim}{y}_1 - \bar{\underset{\sim}{y}} \;\; \underset{\sim}{y}_2 - \bar{\underset{\sim}{y}} \;\; \dots \;\; \underset{\sim}{y}_n - \bar{\underset{\sim}{y}}]^T$$

b

$$E = (Y - X\hat{B})^T (Y - X\hat{B})$$

$$X\hat{B} = [\underset{\sim}{1}_n \; X_2] \begin{bmatrix} \bar{\underset{\sim}{y}}^T \\ (X_2^T X_2)^{-1} X_2^T \tilde{Y} \end{bmatrix} = \underset{\sim}{1}_n \bar{\underset{\sim}{y}}^T + X_2 (X_2^T X_2)^{-1} X_2^T \tilde{Y}$$

$$= \tfrac{1}{n} \underset{\sim}{1}_n \underset{\sim}{1}_n^T Y + M\tilde{Y}, \quad M = X_2 (X_2^T X_2)^{-1} X_2^T$$

$$Y - X\hat{B} = Y - \tfrac{1}{n} \underset{\sim}{1}_n \underset{\sim}{1}_n^T Y - M\tilde{Y} = (I_n - \tfrac{1}{n} \underset{\sim}{1}_n \underset{\sim}{1}_n^T) Y - M\tilde{Y}$$

$$= \tilde{Y} - M\tilde{Y} = (I_n - M)\tilde{Y}$$

$$E = \tilde{Y}^T (I_n - M) \tilde{Y}, \quad \text{as } I_n - M \text{ is symmetric and idempotent}$$

(i) $H_0 : B_2 = 0 \iff [\,\underset{\sim}{0} \;\; I_{p-1}\,] B = 0 \iff AB = 0$

$$A\hat{B} = [\,\underset{\sim}{0} \;\; I_{p-1}\,] \begin{bmatrix} \bar{y}^T \\ (X_2^T X_2)^{-1} X_2^T \tilde{Y} \end{bmatrix} = (X_2^T X_2)^{-1} X_2^T \tilde{Y}$$

$$A(X^TX)^{-1}A^T = [\,\underset{\sim}{0} \;\; I_{p-1}\,] \begin{bmatrix} \frac{1}{n} & \underset{\sim}{0}^T \\ \underset{\sim}{0} & (X_2^T X_2)^{-1} \end{bmatrix} \begin{bmatrix} \underset{\sim}{0}^T \\ I_{p-1} \end{bmatrix} = (X_2^T X_2)^{-1}$$

$$H = (A\hat{B})^T (A(X^TX)^{-1}A^T)^{-1} A\hat{B}$$ cf. "corollary" 1 to theorem 8.5 with $C = 0$

$$= \tilde{Y}^T X_2 (X_2^T X_2)^{-1} X_2^T X_2 (X_2^T X_2)^{-1} X_2^T \tilde{Y}$$
$$= \tilde{Y}^T X_2 (X_2^T X_2)^{-1} X_2^T \tilde{Y}$$
$$= \tilde{Y}^T M \tilde{Y}$$

(ii) $$X B_H = \underset{\sim}{1}_n \underset{\sim}{\beta}_1^T + X_2 0 = [\,\underset{\sim}{1}_n \;\; X_2\,] \begin{bmatrix} \underset{\sim}{\beta}_1^T \\ 0 \end{bmatrix} = \underset{\sim}{1}_n \underset{\sim}{\beta}_1^T$$

$$\hat{B}_H = \begin{bmatrix} \hat{\underset{\sim}{\beta}}_1^T \\ 0 \end{bmatrix} = \begin{bmatrix} (\underset{\sim}{1}_n^T \underset{\sim}{1}_n)^{-1} \underset{\sim}{1}_n^T Y \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{n} \underset{\sim}{1}_n^T Y \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{y}^T \\ 0 \end{bmatrix}$$

$$Y - X\hat{B}_H = Y - [\,\underset{\sim}{1}_n \;\; X_2\,] \begin{bmatrix} \bar{y}^T \\ 0 \end{bmatrix} = Y - \underset{\sim}{1}_n \bar{y}^T = \tilde{Y}$$

$$E_H = (Y - X\hat{B}_H)^T (Y - X\hat{B}_H) = \tilde{Y}^T \tilde{Y}$$

$$H = E_H - E = \tilde{Y}^T \tilde{Y} - \tilde{Y}^T (I_n - M) \tilde{Y} = \tilde{Y}^T M \tilde{Y}$$

(iii) $$P_{\omega^\perp \cap \Omega} = X(X^TX)^{-1}A^T (A(X^TX)^{-1}A^T)^{-1} A (X^TX)^{-1} X^T$$

$$X(X^TX)^{-1}A^T = [\,\underset{\sim}{1}_n \;\; X_2\,] \begin{bmatrix} \frac{1}{n} & \underset{\sim}{0}^T \\ \underset{\sim}{0} & (X_2^T X_2)^{-1} \end{bmatrix} \begin{bmatrix} \underset{\sim}{0} \\ I_{p-1} \end{bmatrix} = X_2 (X_2^T X_2)^{-1}$$

$$P_{\omega^\perp \cap \Omega} = X_2 (X_2^T X_2)^{-1} X_2^T X_2 (X_2^T X_2)^{-1} X_2^T = X_2 (X_2^T X_2)^{-1} X_2 = M$$

$$\tilde{Y} = Y - \tfrac{1}{n} \underset{\sim}{1}_n \underset{\sim}{1}_n^T = Y - [\,\underset{\sim}{1}_n \;\; X_2\,] \begin{bmatrix} \bar{y}^T \\ 0 \end{bmatrix} = Y - XB_0$$

$$AB_0 = [\,\underset{\sim}{0} \;\; I_{p-1}\,] \begin{bmatrix} \bar{y}^T \\ 0 \end{bmatrix} = 0$$

$$H = \tilde{Y}^T P_{\omega^\perp \cap \Omega} \tilde{Y} = \tilde{Y}^T M \tilde{Y}$$

(iv) $\omega = \mathcal{N}(A_1)$, $A_1 = A(X^TX)^{-1}X^T = (X_2^TX_2)^{-1}X_2^T$

$= \mathcal{R}(A_1^T)^\perp = \mathcal{R}(X_2)^\perp = \mathcal{R}(\underset{\sim}{1})$ as $X_2^T\underset{\sim}{1} = \underset{\sim}{0}$

(notice $\mathcal{R}(\underset{\sim}{1}) \subseteq \Omega$)

$$P_\omega = \underset{\sim}{1}(\underset{\sim}{1}^T\underset{\sim}{1})^{-1}\underset{\sim}{1}^T = \frac{1}{n}\underset{\sim}{1}\,\underset{\sim}{1}^T$$

$$P_\Omega - P_\omega = X(X^TX)^{-1}X^T - \frac{1}{n}\underset{\sim}{1}\,\underset{\sim}{1}^T$$

$$X(X^TX)^{-1}X^T = [\underset{\sim}{1}\ X_2]\begin{bmatrix} \frac{1}{n} & \underset{\sim}{0}^T \\ \underset{\sim}{0} & (X_2^TX_2)^{-1} \end{bmatrix}\begin{bmatrix} \underset{\sim}{1}^T \\ X_2^T \end{bmatrix}$$

$$= \frac{1}{n}\underset{\sim}{1}\,\underset{\sim}{1}^T + X_2(X_2^TX_2)^{-1}X_2^T$$

$$P_\Omega - P_\omega = \frac{1}{n}\underset{\sim}{1}\,\underset{\sim}{1}^T + X_2(X_2^TX_2)^{-1}X_2^T - \frac{1}{n}\underset{\sim}{1}\,\underset{\sim}{1}^T = M$$

$$H = \tilde{Y}^T(P_\Omega - P_\omega)\tilde{Y} = \tilde{Y}^TM\tilde{Y}$$

8.8

$X \quad n \times p \quad \text{rank } X = r$

$M \quad q \times n \quad \text{rank } M = q$

$\Omega = \mathcal{R}(X)$

$\omega = \Omega \cap \mathcal{N}(M)$

$P_\omega = P_\Omega - P_{\omega^\perp \cap \Omega}$ cf. B 3.2

$\text{rank } P_{\omega^\perp \cap \Omega} = \text{rank}(P_\Omega M^T)$ cf. B 3.3

$= q$ cf. B 3.4 as $\mathcal{R}(M^T) \cap \Omega^\perp = \underset{\sim}{0}$

indirect proof of $\mathcal{R}(M^T) \cap \Omega^\perp = \underset{\sim}{0}$:

$\exists \underset{\sim}{c} \neq \underset{\sim}{0} : M^T \underset{\sim}{c} \in \Omega^\perp \Rightarrow X^T M^T \underset{\sim}{c} = \underset{\sim}{0} \Leftrightarrow A^T \underset{\sim}{c} = \underset{\sim}{0}$,

rank $A = q$; thus $M^T \underset{\sim}{c} \notin \Omega^\perp$

$\text{rank } P_\omega = \text{rank } P_\Omega - \text{rank } P_{\omega^\perp \cap \Omega} = r - q$

$\text{rank}(I_n - P_\omega) = n - (r - q) = n - r + q$