8.9

The usual linear normal model : $Y = XB + U$

In exercise 8.3 we found the confidence interval for $\underset{\sim}{a}^T \underset{\sim}{\mu}_0 = \underset{\sim}{a}^T B^T \underset{\sim}{x}_0$ as

$$\underset{\sim}{a}^T \underset{\sim}{\mu}_0 = \underset{\sim}{a}^T \hat{\underset{\sim}{y}}_0 \pm t_{1-\frac{\alpha}{2}}(n-p)\sqrt{\underset{\sim}{x}_0^T (X^T X)^{-1} \underset{\sim}{x}_0 \, \underset{\sim}{a}^T S \underset{\sim}{a}}$$

With $\underset{\sim}{a} := \underset{\sim}{b}$ and $\underset{\sim}{x}_0 := \underset{\sim}{a}$ we find $\underset{\sim}{\mu}_0 = B^T \underset{\sim}{a}$ and $\hat{\underset{\sim}{y}}_0 = \hat{B}^T \underset{\sim}{a}$:

$$\underset{\sim}{b}^T B^T \underset{\sim}{a} = \underset{\sim}{b}^T \hat{B}^T \underset{\sim}{a} \pm t_{1-\frac{\alpha}{2}}(n-p)\sqrt{\underset{\sim}{a}^T (X^T X)^{-1} \underset{\sim}{a} \, \underset{\sim}{b}^T S \underset{\sim}{b}}$$

Note that $\underset{\sim}{b}^T B^T \underset{\sim}{a} = \underset{\sim}{a}^T B \underset{\sim}{b}$ and $\underset{\sim}{b}^T \hat{B}^T \underset{\sim}{a} = \underset{\sim}{a}^T \hat{B} \hat{b}$. The confidence interval for $\underset{\sim}{a}^T B \underset{\sim}{b}$ with confidence level $1-\alpha$ then becomes

$$\underset{\sim}{a}^T B \underset{\sim}{b} = \underset{\sim}{a}^T \hat{B} \underset{\sim}{b} \pm t_{1-\frac{\alpha}{2}}(n-p)\sqrt{\underset{\sim}{a}^T (X^T X)^{-1} \underset{\sim}{a} \, \underset{\sim}{b}^T S \underset{\sim}{b}}$$

Considering m pre specified confidence intervals of the type above (i.e. for $\underset{\sim}{a}^T B \underset{\sim}{b}$) with simultaneous confidence level at least $1-\alpha$, we find according to Bonferroni's method that

$$\underset{\sim}{a}^T B \underset{\sim}{b} = \underset{\sim}{a}^T \hat{B} \underset{\sim}{b} \pm t_{1-\frac{\alpha}{2m}}(n-p)\sqrt{\underset{\sim}{a}^T (X^T X)^{-1} \underset{\sim}{a} \, \underset{\sim}{b}^T S \underset{\sim}{b}}$$

The confidence interval found in formula (8.63) was

$$\underset{\sim}{a}^T B \underset{\sim}{b} = \underset{\sim}{a}^T \hat{B} \underset{\sim}{b} \pm \sqrt{(n-p)\, \varphi_{1-\alpha}\, \underset{\sim}{a}^T (X^T X)^{-1} \underset{\sim}{a} \, \underset{\sim}{b}^T S \underset{\sim}{b}}$$

8.10

$$Y = XB + U, \quad \underset{\sim}{u}_i \sim N_d(\underset{\sim}{0}, \Sigma), \quad i = 1, \dots, n, \quad \text{independent}$$

$$X = [X_1 \;\; X_2], \quad X \;\; n \times p, \quad \text{rank } X = p$$

$$X_1 \;\; n \times p_1, \quad X_2 \;\; n \times p_2, \quad p_1 + p_2 = p$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad B \;\; p \times d$$

$$B_1 \;\; p_1 \times d, \quad B_2 \;\; p_2 \times d$$

$$XB = X_1 B_1 + X_2 B_2$$

$$\hat{B} = (X^T X)^{-1} X^T Y$$

$$E = Y^T (I_n - X(X^T X)^{-1} X^T) Y$$

(i) $H_0: \quad B_2 = 0 \iff B_H = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$

$$\hat{B}_H = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} (X_1^T X_1)^{-1} X_1^T Y \\ 0 \end{bmatrix}$$

$$X \hat{B}_H = [X_1 \;\; X_2] \begin{bmatrix} (X_1^T X_1)^{-1} X_1^T Y \\ 0 \end{bmatrix} = X_1 (X_1^T X_1)^{-1} X_1^T Y$$

$$E_H = Y^T (I_n - X_1 (X_1^T X_1)^{-1} X_1^T) Y$$

$$\Lambda = \frac{\det E}{\det(E + H)} = \frac{\det E}{\det E_H} \sim U(d, p_2, n - d) \text{ under } H_0$$

$$H = E_H - E = Y^T (X(X^T X)^{-1} X^T - X_1 (X_1^T X_1) X_1^T) Y$$

(ii) $H_0: \quad B_2 = 0 \iff [0 \;\; I_{p_2}] \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = 0 \iff AB = 0$

$$\omega = \mathcal{N}(A_1), \quad A_1 = A(X^T X)^{-1} X^T = [0 \;\; I_{p_2}] (X^T X)^{-1} X^T$$

$$= \mathcal{R}(A_1^T)^\perp = \mathcal{R}([X_1 \;\; 0]) \quad \text{as} \quad A_1^T [X_1 \;\; 0] = X (X^T X)^{-1} \begin{bmatrix} 0 \\ I_{p_2} \end{bmatrix} [X_1 \;\; 0]$$

$$= 0$$

$$P_\Omega - P_\omega = X (X^T X)^{-1} X^T - X_1 (X_1^T X_1)^{-1} X_1^T$$

$$H = Y^T (P_\Omega - P_\omega) Y = Y^T (X (X^T X)^{-1} X^T - X_1 (X_1^T X_1)^{-1} X_1) Y.$$

thus $E + H$ as well as Λ can be found without first specifying $\hat{B}_H$.

Determination of H from "corollary" 1 to theorem 8.5 and determination of H by use of $P_{\omega^\perp \cap \Omega}$ both involve calculation of $(X^TX)^{-1}A^T(A(X^TX)^{-1}A^T)^{-1}A(X^TX)^{-1}$.

(iii)

$$H = (A\hat{B})^T(A(X^TX)^{-1}A^T)^{-1}A\hat{B}$$

$$= ([0\ I_{p_2}](X^TX)^{-1}X^TY)^T([0\ I_{p_2}](X^TX)^{-1}[0\ I_{p_2}]^T)^{-1}[0\ I_{p_2}](X^TX)^{-1}X^TY$$

$$= Y^TXGX^TY, \quad G = (X^TX)^{-1}\begin{bmatrix}0\\ I_{p_2}\end{bmatrix}\left([0\ I_{p_2}](X^TX)^{-1}\begin{bmatrix}0\\ I_{p_2}\end{bmatrix}\right)^{-1}[0\ I_{p_2}](X^TX)^{-1}$$

Using A3.1 : $\begin{bmatrix}A & B\\ B^T & D\end{bmatrix}^{-1} = \begin{bmatrix}A^{-1}+FE^{-1}F^T & -FE^{-1}\\ -E^{-1}F^T & E^{-1}\end{bmatrix}$, $E = D - B^TA^{-1}B$, $F = A^{-1}B$

Note that

$$\begin{bmatrix}A & B\\ B^T & D\end{bmatrix}^{-1}\begin{bmatrix}0\\ I_{n_2}\end{bmatrix}\left([0\ I_{n_2}]\begin{bmatrix}A & B\\ B^T & D\end{bmatrix}^{-1}\begin{bmatrix}0\\ I_{n_2}\end{bmatrix}\right)^{-1}[0\ I_{n_2}]\begin{bmatrix}A & B\\ B^T & D\end{bmatrix}^{-1}$$

$$= \begin{bmatrix}-FE^{-1}\\ E^{-1}\end{bmatrix}(E^{-1})^{-1}[-E^{-1}F^T\ \ E^{-1}] = \begin{bmatrix}-F\\ I_{n_2}\end{bmatrix}[-E^{-1}F^T\ \ E^{-1}]$$

$$= \begin{bmatrix}FE^{-1}F^T & -FE^{-1}\\ -E^{-1}F^T & E^{-1}\end{bmatrix} = \begin{bmatrix}A^{-1}+FE^{-1}F^T & -FE^{-1}\\ -E^{-1}F^T & E^{-1}\end{bmatrix} - \begin{bmatrix}A^{-1} & 0\\ 0 & 0\end{bmatrix}$$

$$= \begin{bmatrix}A & B\\ B^T & D\end{bmatrix}^{-1} - \begin{bmatrix}A^{-1} & 0\\ 0 & 0\end{bmatrix}$$

Hence $G = (X^TX)^{-1} - \begin{bmatrix}(X_1^TX_1)^{-1} & 0\\ 0 & 0\end{bmatrix}$

$$H = Y^TXGX^TY$$

$$= Y^T\left(X(X^TX)^{-1}X^T - [X_1\ X_2]\begin{bmatrix}(X_1^TX_1)^{-1} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}X_1^T\\ X_2^T\end{bmatrix}\right)Y$$

$$= Y^T(X(X^TX)^{-1}X^T - X_1(X_1^TX_1)^{-1}X_1^T)Y$$

(iv)

$$P_{\omega^\perp\cap\Omega} = X(X^TX)^{-1}A^T(A(X^TX)^{-1}A^T)^{-1}A(X^TX)^{-1}X^T = XGX^T$$

$$= X\left((X^TX)^{-1} - \begin{bmatrix}(X_1^TX_1)^{-1} & 0\\ 0 & 0\end{bmatrix}\right)X^T = X(X^TX)X^T - X_1(X_1^TX_1)^{-1}X_1$$

$$H = Y^TP_{\omega^\perp\cap\Omega}Y = Y^T(X(X^TX)^{-1}X^T - X_1(X_1^TX_1)^{-1}X_1^T)Y$$

8.11

$$H_{01}: \; A_1 B C_1^T = 0 \iff C_1(\underset{\sim}{\mu}_{k-1} - \underset{\sim}{\mu}_k) = \underset{\sim}{0}, \quad k = 2, \dots, K$$

$$H_{03}: \; \underset{\sim}{1}_K^T B C_1^T = \underset{\sim}{0}^T \iff C_1 \sum_{k=1}^{K} \underset{\sim}{\mu}_k = \underset{\sim}{0}$$

$$B = \begin{bmatrix} \underset{\sim}{\mu}_1^T \\ \underset{\sim}{\mu}_2^T \\ \vdots \\ \underset{\sim}{\mu}_K^T \end{bmatrix}, \quad \left.\begin{matrix} A_1 \\ C_1 \end{matrix}\right\} = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \dots & -1 \end{bmatrix} \qquad \begin{matrix} A_1 : (K-1) \times K \\ C_1 : (d-1) \times d \\ \text{rank } A_1 = K-1 \\ \text{rank } C_1 = d-1 \end{matrix}$$

$$\underset{\sim}{x}_i^k \sim N_d(\underset{\sim}{\mu}_k, \Sigma), \quad k = 1, \dots, K, \quad i = 1, \dots, n_k, \quad \text{independent}$$

$$\overline{\underset{\sim}{x}^k} = \frac{1}{n_k} \sum_{i=1}^{n_k} \underset{\sim}{x}_i^k, \quad k = 1, \dots, K$$

$$S_k = \frac{1}{n_k - 1} Q_k, \quad Q_k = \sum_{i=1}^{n_k} (\underset{\sim}{x}_i^k - \overline{\underset{\sim}{x}^k})(\underset{\sim}{x}_i^k - \overline{\underset{\sim}{x}^k})^T, \quad k = 1, \dots, K, \quad \text{indep.}$$

$$\left.\begin{matrix} \overline{\underset{\sim}{x}^k} \sim N_d\left(\underset{\sim}{\mu}_k, \frac{\Sigma}{n_k}\right), \quad k = 1, \dots, K \\ (n_k - 1) S_k \sim W_d(n_k - 1, \Sigma), \quad k = 1, \dots, K \end{matrix}\right\} \text{independent}$$

$$\overline{\underset{\sim}{x}} = \frac{1}{n} \sum_{k=1}^{K} n_k \overline{\underset{\sim}{x}^k}, \quad n = \sum_{k=1}^{K} n_k$$

$$S_p = \frac{1}{n-K} \sum_{k=1}^{K} (n_k - 1) S_k$$

$$\left.\begin{matrix} \overline{\underset{\sim}{x}} \sim N_d\left(\frac{1}{n} \sum_{k=1}^{K} n_k \underset{\sim}{\mu}_k, \frac{\Sigma}{n}\right) \\ (n-K) S_p \sim W_d(n-K, \Sigma) \end{matrix}\right\} \text{independent}$$

$$\left.\begin{matrix} C_1 \overline{\underset{\sim}{x}} \sim N_{d-1}\left(\frac{1}{n} \sum_{k=1}^{K} n_k C_1 \underset{\sim}{\mu}_k, \frac{1}{n} C_1 \Sigma C_1^T\right) \\ (n-K) S_p \sim W_{d-1}(n-K, C_1 \Sigma C_1^T) \end{matrix}\right\} \text{independent}$$

Note that $C_1 \underset{\sim}{\mu}_{k-1} = C_1 \underset{\sim}{\mu}_k, \; k = 2, \dots, K \Rightarrow C_1 \underset{\sim}{\mu}_k = \frac{1}{K} \sum_{k=1}^{K} C_1 \underset{\sim}{\mu}_k, \; k = 1, \dots, K$

$$C_1 \overline{\underset{\sim}{x}} \sim N_{d-1}\left(\frac{1}{K} \sum_{k=1}^{K} C_1 \underset{\sim}{\mu}_k, \frac{1}{n} C_1 \Sigma C_1^T\right) \quad \text{when } H_{01} \text{ is true}$$

$$C_1 \overline{\underset{\sim}{x}} \sim N_{d-1}\left(\underset{\sim}{0}, \frac{1}{n} C_1 \Sigma C_1^T\right) \quad \text{when } H_{01} \cap H_{03} \text{ is true}$$

$$\begin{aligned} T^2 &= n(n-K)(C_1 \overline{\underset{\sim}{x}})^T \left((n-K) C_1 S_p C_1^T\right)^{-1} C_1 \overline{\underset{\sim}{x}} \\ &= n\, \overline{\underset{\sim}{x}}^T C_1^T (C_1 S_p C_1^T)^{-1} C_1 \overline{\underset{\sim}{x}} \\ &\sim T^2(d-1, n-K) \quad \text{when } H_{01} \cap H_{03} \text{ is true, cf. corollary to th. 2.8} \end{aligned}$$

$$[\,y^{(1)}\ \ y^{(2)}\,] = [\,X_1\beta^{(1)}\ \ X_2\beta^{(2)}\,] + [\,u^{(1)}\ \ u^{(2)}\,]$$

$$= [X_1\ X_2]\begin{bmatrix}\beta^{(1)}\\ \beta^{(2)}\end{bmatrix} + [\,u^{(1)}\ \ u^{(2)}\,]$$

$$= X\beta + U$$

$$X = \begin{bmatrix} 1 & x_1 & 1 & x_1 & x_1^2 \\ 1 & x_2 & 1 & x_2 & x_2^2 \\ \vdots & & & & \\ 1 & x_n & 1 & x_n & x_n^2 \end{bmatrix}, \quad \beta^T = [\beta_{10}\ \beta_{11}\ \beta_{20}\ \beta_{21}\ \beta_{22}]$$

$n \geq 3$

$H_0:\ A\beta = 0, \quad A\ q\times 5, \quad q \leq 5$

Presuming that at least three of the x_i's are different, then rank $X_2 = 3$, hence rank $X = 3$. The rows of X therefore span $\mathbb{R}^3$. Thus if only $a_{j1} = a_{j3} \wedge a_{j2} = a_{j4}$, $j = 1, \dots, q$, then $(a^T_j)^T\beta$, $j = 1, \dots, q$, are estimable.

a $\quad H_0:\ \beta_{11} = \beta_{21} = \beta_{22} = 0$

$\Leftrightarrow\ H_{01}:\ \beta_{11} = 0 \ \wedge\ H_{02}:\ \beta_{21} = \beta_{22} = 0$

$\Leftrightarrow\ H_{01}:\ [0\ 1]\beta^{(1)} = 0 \ \wedge\ H_{02}:\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\beta^{(2)} = 0$

$\Leftarrow\ H_0:\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\beta = 0$

$(a^T_j)^T\beta$, $j = 1, 2$, are estimable

b $\quad H_0:\ \beta_{22} = 0$

$\Leftarrow\ H_{01}:$ no restrictions $\wedge\ H_{02}:\ \beta_{22} = 0$

$\Leftrightarrow \quad H_{01}: [0 \; 0] \, \underset{\sim}{\beta}^{(1)} = 0 \quad \wedge \quad H_{02}: [0 \; 0 \; 1] \, \underset{\sim}{\beta}^{(2)} = 0$

$\Leftrightarrow \quad H_0: [0 \; 0 \; 0 \; 0 \; 1] \, \underset{\sim}{\beta} = 0$

$A\underset{\sim}{\beta}$ is estimable

c $\quad H_0: \beta_{11} = \beta_{21} = 0$

$\Leftrightarrow \quad H_{01}: \beta_{11} = 0 \quad \wedge \quad H_{02}: \beta_{21} = 0$

$\Leftrightarrow \quad H_{01}: [0 \; 1] \, \underset{\sim}{\beta}^{(1)} = 0 \quad \wedge \quad H_{02}: [0 \; 1 \; 0] \, \underset{\sim}{\beta}^{(2)} = 0$

$\Leftrightarrow \quad H_0: [0 \; 1 \; 0 \; 1 \; 0] \, \underset{\sim}{\beta} = 0$

$A\underset{\sim}{\beta}$ is estimable

d $\quad H_0: \beta_{11} = 0$

$\Leftrightarrow \quad H_{01}: \beta_{11} = 0 \quad \wedge \quad H_{02}:$ no restrictions

$\Leftrightarrow \quad H_{01}: [0 \; 1] \, \underset{\sim}{\beta}^{(1)} = 0 \quad \wedge \quad H_{02}: [0 \; 0 \; 0] \, \underset{\sim}{\beta}^{(2)} = 0$

$\Leftrightarrow \quad H_0: [0 \; 1 \; 0 \; 0 \; 0] \, \underset{\sim}{\beta} = 0$

$A\underset{\sim}{\beta}$ is not estimable, as $a_{12} \neq a_{14}$

e $\quad H_0: \beta_{11} = \beta_{22} = 0$

$\Leftrightarrow \quad H_{01}: \beta_{11} = 0 \quad \wedge \quad H_{02}: \beta_{22} = 0$

$\Leftrightarrow \quad H_{01}: [0 \; 1] \, \underset{\sim}{\beta}^{(1)} = 0 \quad \wedge \quad H_{02}: [0 \; 0 \; 1] \, \underset{\sim}{\beta}^{(2)} = 0$

$\Leftrightarrow \quad H_0: [0 \; 1 \; 0 \; 0 \; 1] \, \underset{\sim}{\beta} = 0$

$A\underset{\sim}{\beta}$ is not estimable, as $a_{12} \neq a_{14}$