1.7 $\underset{\sim}{x}_1, \dots, \underset{\sim}{x}_n$ sample

$$\operatorname{Var} \underset{\sim}{x}_i = \Sigma \quad , \quad i = 1, \dots, n$$

$$\underset{\sim}{\tilde{x}}_r = \underset{\sim}{x}_r - \underset{\sim}{\bar{x}}$$

$$\operatorname{Cov}(\underset{\sim}{\tilde{x}}_r, \underset{\sim}{\tilde{x}}_s) = \operatorname{Cov}(\underset{\sim}{x}_r - \underset{\sim}{\bar{x}}, \underset{\sim}{x}_s - \underset{\sim}{\bar{x}})$$

$$= \operatorname{Cov}(\underset{\sim}{x}_r, \underset{\sim}{x}_s) - \operatorname{Cov}(\underset{\sim}{x}_r, \underset{\sim}{\bar{x}}) - \operatorname{Cov}(\underset{\sim}{\bar{x}}, \underset{\sim}{x}_s) + \operatorname{Cov}(\underset{\sim}{\bar{x}}, \underset{\sim}{\bar{x}})$$

$$\operatorname{Cov}(\underset{\sim}{x}_r, \underset{\sim}{x}_s) = \delta_{rs} \Sigma$$

$$\operatorname{Cov}(\underset{\sim}{x}_r, \underset{\sim}{\bar{x}}) = \operatorname{Cov}(\underset{\sim}{x}_r, \tfrac{1}{n} \sum_i \underset{\sim}{x}_i) = \frac{1}{n} \sum_i \operatorname{Cov}(\underset{\sim}{x}_r, \underset{\sim}{x}_i)$$

$$= \frac{1}{n} (\Sigma + (n-1)\, 0) = \frac{1}{n} \Sigma$$

$$\operatorname{Cov}(\underset{\sim}{\bar{x}}, \underset{\sim}{x}_s) = \operatorname{Cov}(\underset{\sim}{x}_s, \underset{\sim}{\bar{x}}) = \operatorname{Cov}(\underset{\sim}{x}_r, \underset{\sim}{\bar{x}}) = \frac{1}{n} \Sigma$$

$$\operatorname{Cov}(\underset{\sim}{\bar{x}}, \underset{\sim}{\bar{x}}) = \operatorname{Cov}(\tfrac{1}{n} \sum_i \underset{\sim}{x}_i, \tfrac{1}{n} \sum_j \underset{\sim}{x}_j)$$

$$= \frac{1}{n^2} \sum_i \sum_j \operatorname{Cov}(\underset{\sim}{x}_i, \underset{\sim}{x}_j)$$

$$= \frac{1}{n^2} (n \Sigma + n(n-1)\, 0) = \frac{1}{n} \Sigma$$

$$\operatorname{Cov}(\underset{\sim}{\tilde{x}}_r, \underset{\sim}{\tilde{x}}_s) = \delta_{rs} \Sigma - \frac{1}{n} \Sigma - \frac{1}{n} \Sigma + \frac{1}{n} \Sigma$$

$$= \left(\delta_{rs} - \frac{1}{n} \right) \Sigma$$

1.9 $\underset{\sim}{y} \sim N_d(\underset{\sim}{0}, \Sigma)$, A $d \times d$ sym., rank $A = r$

$\Sigma > 0 \iff \exists R : \Sigma = R^T R$, rank $R = d$, cf. A5.3

Let $\underset{\sim}{y} = R^T \underset{\sim}{z} \iff \underset{\sim}{z} = (R^T)^{-1} \underset{\sim}{y}$

$$\underset{\sim}{z} \sim N_d\left((R^T)^{-1} \underset{\sim}{0} , (R^T)^{-1} (R^T R) \left((R^T)^{-1} \right)^T \right)$$

$$= N_d(\underset{\sim}{0}, I_d)$$

$$\underset{\sim}{y}^T A \underset{\sim}{y} = (R^T \underset{\sim}{z})^T A R^T \underset{\sim}{z} = \underset{\sim}{z}^T R A R^T \underset{\sim}{z}$$

Note that RAR^T is symmetric

and rank (RAR^T) = rank $A = r$

a $\underset{\sim}{y}^T A \underset{\sim}{y} \sim \chi^2(r) \iff \underset{\sim}{z}^T R A R^T \underset{\sim}{z} \sim \chi^2(r)$

$\iff RAR^T$ idempotent, cf. A6.5

$\iff RAR^T RAR^T = RAR^T$

$\iff R^{-1} R A (R^T R) A R^T (R^T)^{-1} = R^{-1} R A R^T (R^T)^{-1}$

$\iff A \Sigma A = A$

b $\underset{\sim}{y}^T A \underset{\sim}{y} \sim \sigma^2 \chi^2(r) \iff \frac{1}{\sigma^2} \underset{\sim}{y}^T A \underset{\sim}{y} \sim \chi^2(r)$

$\iff \underset{\sim}{y}^T \left(\frac{1}{\sigma^2} A \right) \underset{\sim}{y} \sim \chi^2(r)$

$\iff \left(\frac{1}{\sigma^2} A \right) \Sigma \left(\frac{1}{\sigma^2} A \right) = \frac{1}{\sigma^2} A$, cf. a

$\iff A \Sigma A = \sigma^2 A$

2.1 $\underset{\sim}{y} \sim N_d(\underset{\sim}{\theta}, \Sigma)$

Let $\underset{\sim}{y} = \Sigma^{\frac{1}{2}} \underset{\sim}{x} \Leftrightarrow \underset{\sim}{x} = \Sigma^{-\frac{1}{2}} \underset{\sim}{y}$

$$\underset{\sim}{x} \sim N_d(\Sigma^{-\frac{1}{2}}\underset{\sim}{\theta}, \Sigma^{-\frac{1}{2}} \Sigma \Sigma^{-\frac{1}{2}}) = N_d(\Sigma^{-\frac{1}{2}}\underset{\sim}{\theta}, I_d)$$

$$\underset{\sim}{y}^T \Sigma^{-1} \underset{\sim}{y} = \underset{\sim}{x}^T \Sigma^{\frac{1}{2}} \Sigma^{-1} \Sigma^{\frac{1}{2}} \underset{\sim}{x} = \underset{\sim}{x}^T \underset{\sim}{x}$$

$$\underset{\sim}{y}^T \Sigma^{-1} \underset{\sim}{y} \sim \chi^2(d; (\Sigma^{-\frac{1}{2}}\underset{\sim}{\theta})^T(\Sigma^{-\frac{1}{2}}\theta))$$
$$= \chi^2(d; \underset{\sim}{\theta}^T \Sigma^{-1} \underset{\sim}{\theta})$$

2.2 $a \Rightarrow b$ $\quad M_{\underset{\sim}{y}}(\underset{\sim}{t}) = \exp(\underset{\sim}{t}^T \underset{\sim}{\theta} + \frac{1}{2}\underset{\sim}{t}^T \Sigma \underset{\sim}{t})$ given

$$\forall \underset{\sim}{l}: M_{\underset{\sim}{l}^T \underset{\sim}{y}}(t) = M_{\underset{\sim}{y}}(\underset{\sim}{l} t) \quad \text{(lemma 1)}$$
$$= \exp((\underset{\sim}{l} t)^T \underset{\sim}{\theta} + \tfrac{1}{2} (\underset{\sim}{l} t)^T \Sigma (\underset{\sim}{l} t))$$
$$= \exp(\underset{\sim}{l}^T \underset{\sim}{\theta}\, t + \tfrac{1}{2} \underset{\sim}{l}^T \Sigma \underset{\sim}{l}\, t^2)$$

$b \Rightarrow a$ $\quad \forall \underset{\sim}{l}: M_{\underset{\sim}{l}^T \underset{\sim}{y}}(t) = \exp(\underset{\sim}{l}^T \underset{\sim}{\theta}\, t + \frac{1}{2} \underset{\sim}{l}^T \Sigma \underset{\sim}{l}\, t^2)$

given

Choose $t = 1$ and $\underset{\sim}{l} = \underset{\sim}{t}$:

$$\forall \underset{\sim}{t}: M_{\underset{\sim}{t}^T \underset{\sim}{y}}(1) = \exp(\underset{\sim}{t}^T \underset{\sim}{\theta} \cdot 1 + \tfrac{1}{2} \underset{\sim}{t}^T \Sigma \underset{\sim}{t} \cdot 1^2)$$

$$\Leftrightarrow M_{\underset{\sim}{y}}(\underset{\sim}{t}) = \exp(\underset{\sim}{t}^T \underset{\sim}{\theta} + \tfrac{1}{2} \underset{\sim}{t}^T \Sigma \underset{\sim}{t})$$

(lemma 1)

2.3 $\quad \underset{\sim}{y} \sim N_d(\underset{\sim}{\theta}, \Sigma) \iff \forall \underset{\sim}{\ell} : \underset{\sim}{\ell}^T \underset{\sim}{y} \sim N(\underset{\sim}{\ell}^T \underset{\sim}{\theta}, \underset{\sim}{\ell}^T \Sigma \underset{\sim}{\ell})$, $\Sigma \geq 0$

ad (i) :

$$\underset{\sim}{\ell}^T (C\underset{\sim}{y}) = (\underset{\sim}{\ell}^T C)\underset{\sim}{y} = \underset{\sim}{\ell}_1^T \underset{\sim}{y} \sim N(\underset{\sim}{\ell}_1^T \underset{\sim}{\theta}, \underset{\sim}{\ell}_1^T \Sigma \underset{\sim}{\ell}_1)$$
$$= N(\underset{\sim}{\ell}^T C \underset{\sim}{\theta}, \underset{\sim}{\ell}^T C \Sigma C^T \underset{\sim}{\ell})$$

$\Rightarrow C\underset{\sim}{y} \sim N_q(C\underset{\sim}{\theta}, C\Sigma C^T)$, $q = \text{rank } C$ *)

$C\Sigma C^T \geq 0$ cf. A4.4

ad (ii) :

$$\forall \underset{\sim}{\ell} : \underset{\sim}{\ell}^T \underset{\sim}{y} \sim N(\underset{\sim}{\ell}\underset{\sim}{\theta}, \underset{\sim}{\ell}^T \Sigma \underset{\sim}{\ell})$$

$$\Rightarrow \forall \underset{\sim}{\ell}_1 : [\underset{\sim}{\ell}_1^T \ \underset{\sim}{0}^T] \underset{\sim}{y} \sim N\left([\underset{\sim}{\ell}_1^T \ \underset{\sim}{0}^T]\underset{\sim}{\theta}, [\underset{\sim}{\ell}_1^T \ \underset{\sim}{0}^T] \Sigma \begin{bmatrix} \underset{\sim}{\ell}_1 \\ \underset{\sim}{0} \end{bmatrix}\right)$$

$$\iff \forall \underset{\sim}{\ell}_1 : \underset{\sim}{\ell}_1^T \underset{\sim}{y}^{(1)} \sim N(\underset{\sim}{\ell}_1^T \underset{\sim}{\theta}^{(1)}, \underset{\sim}{\ell}_1^T \Sigma_{11} \underset{\sim}{\ell}_1)$$

$\iff \underset{\sim}{y}^{(1)} \sim N_d(\underset{\sim}{\theta}^{(1)}, \Sigma_{11})$, $\Sigma_{11} \geq 0$ *)

ad (iii) :

Has been shown in exercise 2.2

ad (iv) :

$\underset{\sim}{\ell} = \begin{bmatrix} \underset{\sim}{\ell}_1 \\ \underset{\sim}{\ell}_2 \end{bmatrix}$, $\quad \underset{\sim}{\ell}_1 \underset{\sim}{y}^{(1)} \sim N(\underset{\sim}{\ell}_1^T \underset{\sim}{\theta}^{(1)}, \underset{\sim}{\ell}_1^T \Sigma_{11} \underset{\sim}{\ell}_1)$ $\quad \underset{\sim}{\ell}_1$ d_1-dim.

$\quad \underset{\sim}{\ell}_2 \underset{\sim}{y}^{(2)} \sim N(\underset{\sim}{\ell}_2^T \underset{\sim}{\theta}^{(2)}, \underset{\sim}{\ell}_2^T \Sigma_{22} \underset{\sim}{\ell}_2)$ $\quad \underset{\sim}{\ell}_2$ d_2-dim.

$\underset{\sim}{y}^{(1)}$ og $\underset{\sim}{y}^{(2)}$ independent

$\iff \forall \underset{\sim}{\ell}_1, \underset{\sim}{\ell}_2 : \underset{\sim}{\ell}_1^T \underset{\sim}{y}^{(1)}$ og $\underset{\sim}{\ell}_2^T \underset{\sim}{y}^{(2)}$ independent

$$\iff \forall \underset{\sim}{\ell}_1, \underset{\sim}{\ell}_2 : \text{Cov}(\underset{\sim}{\ell}_1^T \underset{\sim}{y}^{(1)}, \underset{\sim}{\ell}_2^T \underset{\sim}{y}^{(2)}) = 0$$

$$\iff \forall \underset{\sim}{\ell}_1, \underset{\sim}{\ell}_2 : \underset{\sim}{\ell}_1^T \text{Cov}(\underset{\sim}{y}^{(1)}, \underset{\sim}{y}^{(2)}) \underset{\sim}{\ell}_2 = 0$$

$$\iff \text{Cov}(\underset{\sim}{y}^{(1)}, \underset{\sim}{y}^{(2)}) = 0$$

*) the notation $\underset{\sim}{y} \sim N_\cdot(\cdot,\cdot)$ is used although Var $\underset{\sim}{y}$ may be singular