2.4 $\underset{\sim}{y}$ stoch. vector with $E\underset{\sim}{y} = \underset{\sim}{\theta}$ and $\operatorname{Var} \underset{\sim}{y} = \Sigma$

a $\Sigma > 0$

Indirect : $\exists\, \underset{\sim}{a} \neq \underset{\sim}{0} : \underset{\sim}{a}^T \underset{\sim}{y} = b$

$\Leftrightarrow \exists\, \underset{\sim}{a} \neq \underset{\sim}{0} : \operatorname{Var} \underset{\sim}{a}^T \underset{\sim}{y} = 0$

$\Leftrightarrow \exists\, \underset{\sim}{a} \neq \underset{\sim}{0} : \underset{\sim}{a}^T \Sigma \underset{\sim}{a} = 0$ inconsistency

Hence $\forall\, \underset{\sim}{a} \neq \underset{\sim}{0} : \underset{\sim}{a}^T \underset{\sim}{y} \neq b$

b $\det \Sigma = 0 \Leftrightarrow \Sigma \geq 0$

$\Leftrightarrow \exists\, \underset{\sim}{a} \neq \underset{\sim}{0} : \underset{\sim}{a}^T \Sigma \underset{\sim}{a} = 0$

$\Leftrightarrow \exists\, \underset{\sim}{a} \neq \underset{\sim}{0} : \operatorname{Var} \underset{\sim}{a}^T \underset{\sim}{y} = 0$

$\Leftrightarrow \exists\, \underset{\sim}{a} \neq \underset{\sim}{0} : \underset{\sim}{a}^T \underset{\sim}{y} = b$

Note that $b = \underset{\sim}{a}^T \underset{\sim}{y} \Rightarrow b = \underset{\sim}{a}^T E \underset{\sim}{y} \Rightarrow b = \underset{\sim}{a}^T \underset{\sim}{\theta}$

Hence $\exists\, \underset{\sim}{a} \neq \underset{\sim}{0} : \underset{\sim}{a}^T \underset{\sim}{y} = \underset{\sim}{a}^T \underset{\sim}{\theta}$

$\Leftrightarrow \exists\, \underset{\sim}{a} \neq \underset{\sim}{0} : \underset{\sim}{a}^T (\underset{\sim}{y} - \underset{\sim}{\theta}) = 0$

2.5 $\quad \underset{\sim}{y} = X\underset{\sim}{\beta} + \underset{\sim}{\varepsilon} \qquad X \;\; n \times p \qquad \underset{\sim}{\varepsilon} \sim N_n(\underset{\sim}{0}, \sigma^2 I_n)$

$\operatorname{rank} X = p < n$

residuals $\quad \underset{\sim}{e} = (I_n - P)\underset{\sim}{y}, \quad P = X(X^TX)^{-1}X^T$

a $\quad \operatorname{Var} \underset{\sim}{e} = (I_n - P)\sigma^2 I_n (I_n - P) = \sigma^2(I_n - P)$

$\operatorname{rank}(I_n - P) = n - p < n$

b $\quad \forall \underset{\sim}{\ell} : \underset{\sim}{\ell}^T \underset{\sim}{e} = \underset{\sim}{\ell}^T (I_n - P)\underset{\sim}{y}$

$= ((I_n - P)\underset{\sim}{\ell})^T \underset{\sim}{y} \sim N(0, \sigma^2 \underset{\sim}{\ell}^T (I_n - P)\underset{\sim}{\ell})$

cf. ex. 1.5 b og 2.5 a

$\Rightarrow \underset{\sim}{e}$ is singular multidimensional normal

as $I_n - P \geq 0$.

c $\quad X = [\underset{\sim}{1}_n \;\; \underset{\sim}{x}^{(1)} \;\; \underset{\sim}{x}^{(2)} \;\; \ldots \;\; \underset{\sim}{x}^{(p-1)}]$

$X^T(I_n - P) = X^T(I_n - X(X^TX)^{-1}X^T)$

$= X^T - X^TX(X^TX)^{-1}X^T = X^T - X^T$

$= O^T = [\underset{\sim}{0} \;\; \underset{\sim}{0} \;\; \ldots \;\; \underset{\sim}{0}]^T$

$\Rightarrow \underset{\sim}{1}_n^T (I_n - P) = \underset{\sim}{0}^T$

$\Rightarrow \underset{\sim}{1}_n^T (I_n - P)\underset{\sim}{y} = 0$

$\Rightarrow \underset{\sim}{1}_n^T \underset{\sim}{e} = 0$

2.6 $\quad y_i \sim N_d(\theta_i, \Sigma_i)$, $i = 1, \dots, n$, independent

a $\quad M_{\sum_i a_i y_i}(t) = \prod_i M_{a_i y_i}(t) = \prod_i M_{y_i}(a_i t)$

$$= \prod_i \exp\left(t^T (a_i \theta_i) + \tfrac{1}{2} (a_i t^T) \Sigma_i (a_i t)\right)$$

$$= \exp\left(t^T \sum_i a_i \theta_i + \tfrac{1}{2} \sum_i a_i^2 t^T \Sigma_i t\right)$$

$$= \exp\left(t^T \sum_i a_i \theta_i + \tfrac{1}{2} t^T \left(\sum_i a_i^2 \Sigma_i\right) t\right)$$

$$\Leftrightarrow \quad \sum_i a_i y_i \sim N_d\left(\sum_i a_i \theta_i, \sum_i a_i^2 \Sigma_i\right)$$

b $\quad \forall \ell : \ell^T y_i \sim N(\ell^T \theta_i, \ell^T \Sigma_i \ell)$, $i = 1, \dots, n$

$$\Rightarrow \quad \forall \ell : \sum_i a_i \ell^T y_i \sim N\left(\sum_i a_i \ell^T \theta_i, \sum_i a_i^2 \ell^T \Sigma_i \ell\right)$$

$$\Leftrightarrow \quad \forall \ell : \ell^T \sum_i a_i y_i \sim N\left(\ell^T \sum_i a_i \theta_i, \ell^T \left(\sum_i a_i^2 \Sigma_i\right) \ell\right)$$

$$\Leftrightarrow \quad \sum_i a_i y_i \sim N_d\left(\sum_i a_i \theta_i, \sum_i a_i^2 \Sigma_i\right)$$

2.7 $\quad y_i \sim N_d(0, \Sigma)$, $i = 1, \dots, n$, independent

Sæt $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ og $\ell = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \vdots \\ \ell_n \end{bmatrix}$

$$\forall \ell_i : \ell_i^T y_i \sim N(0, \ell_i^T \Sigma \ell_i) \wedge \mathrm{Cov}(y_i, y_j) = 0$$

$$\Leftrightarrow \quad \forall \ell : \ell^T y \sim N\left(0, \sum_i \ell_i^T \Sigma \ell_i\right)$$

$$\Leftrightarrow \quad \forall \ell : \ell^T y \sim N\left(0, \ell^T (I_n \otimes \Sigma) \ell\right)$$

$$\Leftrightarrow \quad y \sim N_{nd}(0, I_n \otimes \Sigma)$$

2.8 $y_i \sim N(\theta, \sigma^2)$, $i = 1, \dots, n$, independent

$\Leftrightarrow \quad \underset{\sim}{y} = [y_1 \dots y_n]^T \sim N_n(\theta \underset{\sim}{1_n}, \sigma^2 I_n)$

$Q = \sum_i (y_i - \bar{y})^2 = (\underset{\sim}{y} - \bar{y} \underset{\sim}{1_n})^T (\underset{\sim}{y} - \bar{y} \underset{\sim}{1_n})$

a $\quad \bar{y} = \frac{1}{n} \sum_i y_i = \frac{1}{n} \underset{\sim}{1_n}^T \underset{\sim}{y} = A_1 \underset{\sim}{y}$

$y_1 - \bar{y} = ([1 \; 0 \dots 0] - \frac{1}{n} \underset{\sim}{1_n}^T) \underset{\sim}{y} = A_2 \underset{\sim}{y}$

$A_1 A_2^T = \frac{1}{n} \underset{\sim}{1_n}^T ([1 \; 0 \dots 0]^T - \frac{1}{n} \underset{\sim}{1_n}) = \frac{1}{n}(1 - \frac{1}{n} n) = 0$

$\Leftrightarrow$ $\bar{y}$ og $y_1 - \bar{y}$ independent cf. sætn. 2.1 (v)

$\Rightarrow$ $\bar{y}$ og $y_i - \bar{y}$ independent , $i = 1, \dots, n$

$\Rightarrow$ $\bar{y}$ og Q independent

b $\quad Q = (\underset{\sim}{y} - \frac{1}{n} \underset{\sim}{1_n}^T \underset{\sim}{y} \underset{\sim}{1_n})^T (\underset{\sim}{y} - \frac{1}{n} \underset{\sim}{1_n}^T \underset{\sim}{y} \underset{\sim}{1_n})$

$= (\underset{\sim}{y} - \frac{1}{n} \underset{\sim}{1_n} \underset{\sim}{1_n}^T \underset{\sim}{y})^T (\underset{\sim}{y} - \frac{1}{n} \underset{\sim}{1_n} \underset{\sim}{1_n}^T \underset{\sim}{y})$

$= \underset{\sim}{y}^T (I_n - \frac{1}{n} \underset{\sim}{1_n} \underset{\sim}{1_n}^T)^T (I_n - \frac{1}{n} \underset{\sim}{1_n} \underset{\sim}{1_n}^T) \underset{\sim}{y}$

$= \underset{\sim}{y}^T (I_n - \frac{1}{n} \underset{\sim}{1_n} \underset{\sim}{1_n}^T)(I_n - \frac{1}{n} \underset{\sim}{1_n} \underset{\sim}{1_n}^T) \underset{\sim}{y}$

$= \underset{\sim}{y}^T (I_n - \frac{1}{n} \underset{\sim}{1_n} \underset{\sim}{1_n}^T - \frac{1}{n} \underset{\sim}{1_n} \underset{\sim}{1_n}^T + \frac{1}{n^2} \underset{\sim}{1_n} n \underset{\sim}{1_n}^T) \underset{\sim}{y}$

$= \underset{\sim}{y}^T (I_n - \frac{1}{n} \underset{\sim}{1_n} \underset{\sim}{1_n}^T) \underset{\sim}{y} = \underset{\sim}{y}^T A \underset{\sim}{y}$

The calculations show that $A^T = A$ and $A^2 = A$,

also notice rank A = tr A = $n - 1$, cf. A 6.1

Let $u_i = y_i - \theta$, $i = 1, \dots, n$, $u_i \sim N(0, \sigma^2)$ independent

$Q = (\underset{\sim}{u} + \underset{\sim}{\theta})^T A (\underset{\sim}{u} + \underset{\sim}{\theta}) = \underset{\sim}{u}^T A \underset{\sim}{u} + 2 \underset{\sim}{u}^T A \underset{\sim}{\theta} + \underset{\sim}{\theta}^T A \underset{\sim}{\theta}$

Note : $\quad A \underset{\sim}{\theta} = (I_n - \frac{1}{n} \underset{\sim}{1_n} \underset{\sim}{1_n}^T) \theta \underset{\sim}{1_n} = \theta (\underset{\sim}{1_n} - \frac{1}{n} \underset{\sim}{1_n} \underset{\sim}{1_n}^T \underset{\sim}{1_n})$

$= \theta (\underset{\sim}{1_n} - \underset{\sim}{1_n}) = \underset{\sim}{0}$

Hence $\quad Q = \underset{\sim}{y}^T A \underset{\sim}{y} = \underset{\sim}{u}^T A \underset{\sim}{u} \sim \chi^2(n-1)$ cf. A 6.5