2.9 $W \sim W_d(m, \Sigma)$

$$f(\underbrace{w_{11}, \dots, w_{dd}}_{\frac{d(d+1)}{2}\text{ var.}}) = \frac{1}{c} (\det W)^{\frac{m-d-1}{2}} \operatorname{etr}\left(-\tfrac{1}{2}\Sigma^{-1} W\right) ,$$

$$c = 2^{\frac{md}{2}} (\det \Sigma)^{\frac{m}{2}} \Gamma_d\left(\tfrac{m}{2}\right)$$

$$M_W(U) = E\left[\operatorname{etr}(UW)\right]$$

$$= \int_{R^{\frac{d(d+1)}{2}}} \operatorname{etr}(UW) \frac{1}{c} (\det W)^{\frac{m-d-1}{2}} \operatorname{etr}\left(-\tfrac{1}{2}\Sigma^{-1} W\right) d\Omega$$

$$= (\det \Sigma)^{-\frac{m}{2}} \left(\det (\Sigma^{-1} - 2U)^{-1}\right)^{\frac{m}{2}}$$

$$\cdot \int_{R^{\frac{d(d+1)}{2}}} \frac{1}{c_1} (\det W)^{\frac{m-d-1}{2}} \operatorname{etr}\left(-\tfrac{1}{2}(\Sigma^{-1} - 2U) W\right) d\Omega ,$$

$$c_1 = 2^{\frac{md}{2}} \left(\det (\Sigma^{-1} - 2U)^{-1}\right)^{\frac{m}{2}} \Gamma_d\left(\tfrac{m}{2}\right)$$

$$= (\det \Sigma)^{-\frac{m}{2}} \left(\det(\Sigma^{-1} - 2U)\right)^{-\frac{m}{2}} \cdot 1$$

$$= \left(\det(I_d - 2\Sigma U)\right)^{-\frac{m}{2}}$$

$I_d - 2U\Sigma > 0$ in a neighbourhood of $U = 0$, cf A 5.8

$\Rightarrow$ $M_W(U)$ exists in a neighbourhood of $U = 0$

$\Rightarrow$ $M_W(U)$ uniquely determines the joint density function

2.10 Let $W = \sum_{i=1}^{m} \underset{\sim}{x_i} \underset{\sim}{x_i}^T = X^T X$, $\underset{\sim}{x_i} \sim N(\underset{\sim}{0}, \Sigma)$, $i = 1, \dots, m$ independent

1st method: $EW = E\left[X^T X\right] = E\left[X^T I_m X\right]$

$$= (\operatorname{tr} I_m) \operatorname{Var} X + (EX)^T I_m EX \quad \text{cf. formula (1.19)}$$

$$= m\Sigma + O^T I_m O = m\Sigma$$

2nd method: $EW = \sum_{i=1}^{m} E\left[\underset{\sim}{x_i} \underset{\sim}{x_i}^T\right] = \sum_{i=1}^{m} \left(\operatorname{Var} \underset{\sim}{x_i} + E\underset{\sim}{x_i}\, E\underset{\sim}{x_i}^T\right)$

$$= \sum_{i=1}^{m} \left(\Sigma + \underset{\sim}{0}\,\underset{\sim}{0}^T\right) = \sum_{i=1}^{m} (\Sigma + O) = m\Sigma$$

2.11

i $\quad f(w_{11}, \dots, w_{dd}) = \frac{1}{c} (\det W)^{\frac{m-d-1}{2}} \operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1} W\right),$

$$c = 2^{\frac{md}{2}} (\det \Sigma)^{\frac{m}{2}} \Gamma_d\left(\frac{m}{2}\right),$$

$$\Gamma_d\left(\frac{m}{2}\right) = \pi^{\frac{d(d-1)}{4}} \prod_{j=1}^{d} \Gamma\left(\frac{m+1-j}{2}\right)$$

$d = 1$ and $w_{11} = w$:

$$f(w) = \frac{1}{c} w^{\frac{m-2}{2}} e^{-\frac{w}{2\sigma^2}}, \quad c = 2^{\frac{m}{2}} (\sigma^2)^{\frac{m}{2}} \Gamma\left(\frac{m}{2}\right)$$

$$= \frac{1}{\Gamma(\frac{m}{2})} (2\sigma^2)^{-\frac{m}{2}} w^{\frac{m-2}{2}} e^{-\frac{w}{2\sigma^2}}, \quad w > 0 \qquad *$$

$$\Leftrightarrow \quad w \sim \sigma^2 \chi^2(m)$$

ii $\quad W = \sum_{i=1}^{m} \underset{\sim}{x}_i \underset{\sim}{x}_i^T, \quad \underset{\sim}{x}_i \sim N_d(\underset{\sim}{0}, \Sigma), \quad i = 1, \dots, m$
independent

$d = 1$:

$$w = \sum_{i=1}^{m} x_i^2, \quad x_i \sim N(0, \sigma^2), \quad i = 1, \dots, m$$
independent

$$\Rightarrow \quad w \sim \sigma^2 \chi^2(m)$$

$* \quad v \sim \chi^2(m) \Leftrightarrow f(v) = \frac{1}{\Gamma(\frac{m}{2})} 2^{-\frac{m}{2}} v^{\frac{m-2}{2}} e^{-\frac{v}{2}}, \quad v > 0$

Let $w = \sigma^2 v \Leftrightarrow v = \frac{w}{\sigma^2}, \quad \frac{dv}{dw} = \frac{1}{\sigma^2}$

$$f(w) = \frac{1}{\Gamma(\frac{m}{2})} 2^{-\frac{m}{2}} \left(\frac{w}{\sigma^2}\right)^{\frac{m-2}{2}} e^{-\frac{w}{2\sigma^2}} \left|\frac{1}{\sigma^2}\right|$$

$$= \frac{1}{\Gamma(\frac{m}{2})} 2^{-\frac{m}{2}} (\sigma^2)^{-\frac{m-2}{2}-1} w^{\frac{m-2}{2}} e^{-\frac{w}{2\sigma^2}}$$

$$= \frac{1}{\Gamma(\frac{m}{2})} (2\sigma^2)^{-\frac{m}{2}} w^{\frac{m-2}{2}} e^{-\frac{w}{2\sigma^2}}, \quad w > 0$$

2.12 $\quad W \sim W_d(m, \Sigma) \quad , \quad \Sigma > 0$

a $\quad M_{\underset{\sim}{\ell}^T W \underset{\sim}{\ell}}(t) = E[\exp(t \underset{\sim}{\ell}^T W \underset{\sim}{\ell})] = E[\mathrm{etr}(t \underset{\sim}{\ell}^T W \underset{\sim}{\ell})]$

$= E[\mathrm{etr}(t \underset{\sim}{\ell} \underset{\sim}{\ell}^T W)]$ note that $t \underset{\sim}{\ell} \underset{\sim}{\ell}^T$ is sym.

$= (\det(I_d - 2t \underset{\sim}{\ell} \underset{\sim}{\ell}^T \Sigma))^{-\frac{m}{2}}$ cf. ex. 2.9

$= (\det(I_1 - 2t \underset{\sim}{\ell}^T \Sigma \underset{\sim}{\ell}))^{-\frac{m}{2}}$ cf. hint in the text

$= (1 - 2t\sigma_\ell^2)^{-\frac{m}{2}} \quad , \quad \sigma_\ell^2 = \underset{\sim}{\ell}^T \Sigma \underset{\sim}{\ell}$

$= (1 - 2\sigma_\ell^2 t)^{-\frac{m}{2}}$

$\Leftrightarrow \quad \underset{\sim}{\ell}^T W \underset{\sim}{\ell} \sim \sigma_\ell^2 \chi^2(m)$

b $\quad W = \begin{array}{cc} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} & \begin{array}{c} r \\ d-r \end{array} \end{array} \quad , \quad \Sigma = \begin{array}{cc} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} & \begin{array}{c} r \\ d-r \end{array} \end{array}$

$M_{W_{11}}(U_{11}) = M_{\begin{bmatrix} W_{11} & 0 \\ 0 & 0 \end{bmatrix}}\left(\begin{bmatrix} U_{11} & 0 \\ 0 & 0 \end{bmatrix}\right)$

$= \left(\det\left(I_d - 2\begin{bmatrix} U_{11} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right)\right)^{-\frac{m}{2}}$ cf. ex. 2.9

$= \left(\det\left(I_d - 2\begin{bmatrix} U_{11}\Sigma_{11} & U_{11}\Sigma_{12} \\ 0 & 0 \end{bmatrix}\right)\right)^{-\frac{m}{2}}$

$= \left(\det\begin{bmatrix} I_r - 2U_{11}\Sigma_{11} & -2U_{11}\Sigma_{12} \\ 0 & I_{d-r} \end{bmatrix}\right)^{-\frac{m}{2}}$

$= (\det(I_r - 2U_{11}\Sigma_{11}) \det I_{d-r})^{-\frac{m}{2}}$

$= (\det(I_r - 2U_{11}\Sigma_{11}))^{-\frac{m}{2}}$

$\Leftrightarrow \quad W_{11} \sim W_r(m, \Sigma_{11})$

In the same way $\quad W_{22} \sim W_{d-r}(m, \Sigma_{22})$

$M_{W_{11}+W_{22}}(U_{11}, U_{22}) = M_{\begin{bmatrix} W_{11} & 0 \\ 0 & W_{22} \end{bmatrix}}\left(\begin{bmatrix} U_{11} & 0 \\ 0 & U_{22} \end{bmatrix}\right)$

$= \left(\det\left(I_d - 2\begin{bmatrix} U_{11} & 0 \\ 0 & U_{22} \end{bmatrix}\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\right)\right)^{-\frac{m}{2}}$

$= \left(\det\left(I_d - 2\begin{bmatrix} U_{11}\Sigma_{11} & U_{11}\Sigma_{12} \\ U_{22}\Sigma_{21} & U_{22}\Sigma_{22} \end{bmatrix}\right)\right)^{-\frac{m}{2}}$

$$\Sigma_{12} = 0 \Rightarrow M_{W_{11}+W_{22}}(U_{11}, U_{22}) = \left(\det\left(I_d - 2\begin{bmatrix} U_{11}\Sigma_{11} & 0 \\ 0 & U_{22}\Sigma_{22}\end{bmatrix}\right)\right)^{-\frac{m}{2}}$$

$$= \left(\det\begin{bmatrix} I_r - 2U_{11}\Sigma_{11} & 0 \\ 0 & I_{d-r} - 2U_{22}\Sigma_{22}\end{bmatrix}\right)^{-\frac{m}{2}}$$

$$= \left(\det(I_r - 2U_{11}\Sigma_{11})\right)^{-\frac{m}{2}} \left(\det(I_{d-r} - 2U_{22}\Sigma_{22})\right)^{-\frac{m}{2}}$$

$\Leftrightarrow$ W_{11} and W_{22} independent

c Σ's eigenvalues are called λ_i, $i = 1, \dots, d$

$$M_{\operatorname{tr} W}(t) = E[\exp(t \operatorname{tr} W)] = E[\operatorname{etr}(tW)] = E[\operatorname{etr}(t I_d W)]$$

$$= (\det(I_d - 2t I_d \Sigma))^{-\frac{m}{2}} = (\det(I_d - 2t\Sigma))^{-\frac{m}{2}}$$

$$= \left(\prod_{i=1}^{d}(1 - 2t\lambda_i)\right)^{-\frac{m}{2}} \quad \text{vgl. A 1.2 c}$$

$$= \prod_{i=1}^{d}(1 - 2\lambda_i t)^{-\frac{m}{2}} = \prod_{i=1}^{d} M_{\lambda_i v_i}(t) = M_{\sum_{i=1}^{d}\lambda_i v_i}(t),$$

where $v_i \sim \chi^2(m)$, $i = 1, \dots, d$, independent

$$\Leftrightarrow \operatorname{tr} W \sim \sum_{i=1}^{d} \lambda_i v_i$$

d $\left.\begin{array}{l} W_1 \sim W_d(m_1, \Sigma) \\ W_2 \sim W_d(m_2, \Sigma) \end{array}\right\}$ independent

$$M_{W_1+W_2}(U) = M_{W_1}(U)\, M_{W_2}(U)$$

$$= E[\operatorname{etr}(UW_1)]\; E[\operatorname{etr}(UW_2)]$$

$$= (\det(I_d - 2U\Sigma))^{-\frac{m_1}{2}} (\det(I_d - 2U\Sigma))^{-\frac{m_2}{2}}$$

$$= (\det(I_d - 2U\Sigma))^{-\frac{m_1+m_2}{2}}$$

$$\Leftrightarrow W_1 + W_2 \sim W_d(m_1 + m_2, \Sigma)$$