2.13 $\left.\begin{array}{l} W_1 \sim W_d(m_1, \Sigma) \\ W_2 \sim W_d(m_2, \Sigma) \end{array}\right\}$ independant

$$\Leftrightarrow \left\{ \begin{array}{llll} W_1 = \sum_{i=1}^{m_1} \underset{\sim}{x_i} \underset{\sim}{x_i}^T, & \underset{\sim}{x_i} \sim N_d(\underset{\sim}{0}, \Sigma), & i = 1, \dots, m_1 \\ W_2 = \sum_{i=m_1+1}^{m_1+m_2} \underset{\sim}{x_i} \underset{\sim}{x_i}^T, & x_i \sim N_d(\underset{\sim}{0}, \Sigma), & i = m_1+1, \dots, m_1+m_2 \end{array} \right\},$$

$\underset{\sim}{x_i}$'s independent

$$\Rightarrow \quad W_1 + W_2 = \sum_{i=1}^{m_1+m_2} \underset{\sim}{x_i} \underset{\sim}{x_i}^T, \quad \underset{\sim}{x_i} \sim N_d(\underset{\sim}{0}, \Sigma), \quad i = 1, \dots, m_1+m_2,$$

independent

$$\Leftrightarrow \quad W_1 + W_2 \sim W_d(m_1+m_2, \Sigma)$$

2.14 $W \sim W_d(m, \Sigma)$

$$W = \sum_i \underset{\sim}{x_i} \underset{\sim}{x_i}^T, \quad \underset{\sim}{x_i} \sim N_d(\underset{\sim}{0}, \Sigma), \quad i = 1, \dots, m$$

$$\underset{\sim}{x_i} = \begin{bmatrix} \underset{\sim}{x_i}^{(1)} \\ \underset{\sim}{x_i}^{(2)} \end{bmatrix} \begin{array}{l} r\text{-dim.} \\ (d-r)\text{-dim.} \end{array} \qquad \Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{array}{l} r \\ d-r \end{array}$$

$$W = \sum_i \begin{bmatrix} \underset{\sim}{x_i}^{(1)} \\ \underset{\sim}{x_i}^{(2)} \end{bmatrix} \begin{bmatrix} \underset{\sim}{x_i}^{(1)T} & \underset{\sim}{x_i}^{(2)T} \end{bmatrix} = \begin{bmatrix} \sum_i \underset{\sim}{x_i}^{(1)} \underset{\sim}{x_i}^{(1)T} & \sum_i \underset{\sim}{x_i}^{(1)} \underset{\sim}{x_i}^{(2)T} \\ \sum_i \underset{\sim}{x_i}^{(2)} \underset{\sim}{x_i}^{(1)T} & \sum_i \underset{\sim}{x_i}^{(2)} \underset{\sim}{x_i}^{(2)T} \end{bmatrix}$$

$$= \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

$\left.\begin{array}{l} W_{11} \sim W_d(m, \Sigma_{11}) \\ W_{22} \sim W_d(m, \Sigma_{22}) \end{array}\right\}$ cf. th. 2.1 (ii) and def. 2b

W_{11} og W_{22} are in general dependent

($\Sigma_{12} = 0$ secure independency, cf. ex. 2.12 b)

2.15 $\quad y \sim N_d(\underset{\sim}{0}, \Sigma) \quad , \quad A \quad d \times d$

$$M_{y^T A y}(t) = E\left[\exp(t\, y^T A y)\right]$$

$$= \int_{R^d} \exp(t\, y^T A y)\,(2\pi)^{-\frac{d}{2}}(\det \Sigma)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2} y^T \Sigma y\right) d\Omega$$

$$= \int_{R^d} (2\pi)^{-\frac{d}{2}}(\det \Sigma)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2} y^T(\Sigma^{-1} - 2tA) y\right) d\Omega$$

$$= (\det(I_d - 2tA\Sigma))^{-\frac{1}{2}} \int_{R^d} (2\pi)^{-\frac{d}{2}}(\det \Sigma)^{-\frac{1}{2}}(\det(I_d - 2tA\Sigma))^{\frac{1}{2}} \cdot$$
$$\cdot \exp\left(-\tfrac{1}{2} y^T(\Sigma^{-1} - 2tA) y\right) d\Omega$$

$$= (\det(I_d - 2tA\Sigma))^{-\frac{1}{2}} \int_{R^d} (2\pi)^{-\frac{d}{2}}(\det(\Sigma^{-1} - 2tA)^{-1})^{-\frac{1}{2}} \cdot$$
$$\cdot \exp\left(-\tfrac{1}{2} y^T(\Sigma^{-1} - 2tA) y\right) d\Omega$$

$$= (\det(I_d - 2tA\Sigma))^{-\frac{1}{2}} \cdot 1 \qquad *$$

$$= (\det(I_d - 2tA\Sigma))^{-\frac{1}{2}}$$

$*$ $\quad \Sigma > 0 \Rightarrow \Sigma^{-1} > 0 \Rightarrow \Sigma^{-1} - 2tA > 0$ in a neighbourhood of $t=0$

$\Rightarrow (\Sigma^{-1} - 2tA)^{-1} > 0$ in a neigbourhood of $t=0$

2.16 $\quad \underset{\sim}{x_i} \sim N_d(\underset{\sim}{0}, \Sigma) \quad , \quad i = 1, \dots, m \quad ,$ independent , $m \geq d$

$$W = \sum_i \underset{\sim}{x_i}\underset{\sim}{x_i}^T \sim W_d(m, \Sigma)$$

$$M_W(U) = E\left[\operatorname{etr}(UW)\right] = E\left[\operatorname{etr}\left(U \sum_i \underset{\sim}{x_i}\underset{\sim}{x_i}^T\right)\right]$$

$$= E\left[\operatorname{etr} \sum_i U \underset{\sim}{x_i}\underset{\sim}{x_i}^T\right] = E\left[\operatorname{etr} \sum_i \underset{\sim}{x_i}^T U \underset{\sim}{x_i}\right]$$

$$= E\left[\exp \sum_i \underset{\sim}{x_i}^T U \underset{\sim}{x_i}\right] = \prod_i E\left[\exp(\underset{\sim}{x_i}^T U \underset{\sim}{x_i})\right]$$

$$= \prod_i (\det(I_d - 2U\Sigma))^{-\frac{1}{2}} \qquad \text{cf. ex. 2.15 with } t = 1$$

$$= (\det(I_d - 2U\Sigma))^{-\frac{m}{2}}$$