2.18

$$\underset{\sim}{x}_i \sim N_d(\underset{\sim}{0}, \Sigma) \quad , \quad i = 1, \dots, n \quad , \quad \text{independent}$$

$$X = [\underset{\sim}{x}_1 \dots \underset{\sim}{x}_n]^T$$

$$\left.\begin{array}{l} Y = AXB \\ Z = CXD \end{array}\right\} \quad A, B, C, D \text{ matrices with constant elements}$$

$$y_{ij} = \sum_k \sum_l a_{ik} x_{kl} b_{lj} \quad , \quad z_{rs} = \sum_p \sum_q c_{rp} x_{pq} d_{qs}$$

$$\begin{aligned}
\mathrm{Cov}(y_{ij}, z_{rs}) &= \sum_k \sum_p a_{ik} c_{rp} \mathrm{Cov}\Big(\sum_l x_{kl} b_{lj}, \sum_q x_{pq} d_{qs}\Big) \\
&= \sum_k \sum_p a_{ik} c_{rp} \sum_l \sum_q b_{lj} d_{qs} \mathrm{Cov}(x_{kl}, x_{pq}) \\
&= \sum_k a_{ik} c_{rk} \sum_l \sum_q b_{lj} d_{qs} \mathrm{Cov}(x_{kl}, x_{kq}) \\
&= \sum_k a_{ik} c^T_{kr} \sum_l b_{lj} \sum_q \sigma_{lq} d_{qs} \\
&= (AC^T)_{ir} \sum_l b^T_{jl} (\Sigma D)_{ls} \\
&= (AC^T)_{ir} (B^T \Sigma D)_{js}
\end{aligned}$$

y_{ij} og z_{rs} independent for all (i,j) and (r,s)

$\Leftrightarrow$ $\mathrm{Cov}(y_{ij}, z_{rs}) = 0$ for all (i,j) and (r,s)

$\Leftrightarrow$ $(AC^T)_{ir} = 0 \vee (B^T \Sigma D)_{js} = 0$ for all (i,j) and (r,s)

$\Leftrightarrow$ $AC^T = 0 \vee B^T \Sigma D = 0$

Notice that specific cases of the rule appear when some of the matrices A, B, C or D are identity matrices.

2.19 Assume that $f(x|y)$ does not depend on y

$$f_x(x) = \int f(x,y)\,dy = \int f(x|y) f_y(y)\,dy = f(x|y)\int f_y(y)\,dy$$
$$= f(x|y)\cdot 1 = f(x|y)$$

$$f(x,y) = f(x|y) f_y(y) = f_x(x) f_y(y) \Leftrightarrow x \text{ and } y \text{ independent}$$

2.20 $\Sigma > 0$

$$\underset{\sim}{\mu}^T \Sigma^{-1} \underset{\sim}{\mu}$$

$$= [\underset{\sim}{\mu}_1^T \ \underset{\sim}{\mu}_2^T] \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}^{-1} \begin{bmatrix} \underset{\sim}{\mu}_1 \\ \underset{\sim}{\mu}_2 \end{bmatrix}, \quad \text{sæt } E = \Sigma_{22\cdot1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \text{ og } F = \Sigma_{11}^{-1}\Sigma_{12}$$

$$= [\underset{\sim}{\mu}_1^T \ \underset{\sim}{\mu}_2^T] \begin{bmatrix} \Sigma_{11}^{-1} + FE^{-1}F^T & -FE^{-1} \\ -E^{-1}F^T & E^{-1} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\mu}_1 \\ \underset{\sim}{\mu}_2 \end{bmatrix}, \quad \text{cf. A3.1}$$

$$= [\underset{\sim}{\mu}_1^T \Sigma_{11}^{-1} + \underset{\sim}{\mu}_1^T FE^{-1}F^T - \underset{\sim}{\mu}_2^T E^{-1}F^T \quad -\underset{\sim}{\mu}_1^T FE^{-1} + \underset{\sim}{\mu}_2^T E^{-1}] \begin{bmatrix} \underset{\sim}{\mu}_1 \\ \underset{\sim}{\mu}_2 \end{bmatrix}$$

$$= \underset{\sim}{\mu}_1^T \Sigma_{11}^{-1} \underset{\sim}{\mu}_1 + \underset{\sim}{\mu}_1^T FE^{-1}F^T \underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_2^T E^{-1}F^T \underset{\sim}{\mu}_1 - \underset{\sim}{\mu}_1^T FE^{-1} \underset{\sim}{\mu}_2 + \underset{\sim}{\mu}_2^T E^{-1} \underset{\sim}{\mu}_2$$

$$= \underset{\sim}{\mu}_1^T \Sigma_{11}^{-1} \underset{\sim}{\mu}_1 + (\underset{\sim}{\mu}_2 - F^T \underset{\sim}{\mu}_1)^T E^{-1} (\underset{\sim}{\mu}_2 - F^T \underset{\sim}{\mu}_1)$$

$$= \underset{\sim}{\mu}_1^T \Sigma_{11}^{-1} \underset{\sim}{\mu}_1 + (\underset{\sim}{\mu}_2 - \Sigma_{21}\Sigma_{11}^{-1} \underset{\sim}{\mu}_1)^T \Sigma_{22\cdot1}^{-1} (\underset{\sim}{\mu}_2 - \Sigma_{21}\Sigma_{11}^{-1} \underset{\sim}{\mu}_1)$$

$$= \underset{\sim}{\mu}_1^T \Sigma_{11}^{-1} \underset{\sim}{\mu}_1 + \underset{\sim}{\mu}_{2\cdot1}^T \Sigma_{22\cdot1}^{-1} \underset{\sim}{\mu}_{2\cdot1}, \quad \text{where } \underset{\sim}{\mu}_{2\cdot1} = \underset{\sim}{\mu}_2 - \Sigma_{21}\Sigma_{11}^{-1} \underset{\sim}{\mu}_1$$

$$\Leftrightarrow \underset{\sim}{\mu}^T \Sigma^{-1} \underset{\sim}{\mu} - \underset{\sim}{\mu}_1^T \Sigma_{11}^{-1} \underset{\sim}{\mu}_1 = \underset{\sim}{\mu}_{2\cdot1}^T \Sigma_{22\cdot1}^{-1} \underset{\sim}{\mu}_{2\cdot1}$$

$\Sigma > 0 \Rightarrow \Sigma^{-1} > 0$ cf. A5.5

$\Rightarrow \Sigma_{22\cdot1}^{-1} > 0$ cf. A5.9 modified (leading minor determinants are taken from "the opposite end")

$\Rightarrow \Sigma_{22\cdot1} > 0$ cf. A5.5

2.21

$$\left.\begin{array}{l} W_1 \sim W_d(m_1, \Sigma) \\ W_2 \sim W_d(m_2, \Sigma) \end{array}\right\} \text{ independent } *$$

$$\frac{\det W_1}{\det(W_1+W_2)} = \frac{\det \Sigma^{-\frac{1}{2}} \det W_1 \det \Sigma^{-\frac{1}{2}}}{\det \Sigma^{-\frac{1}{2}} \det(W_1+W_2) \det \Sigma^{-\frac{1}{2}}}$$

$$= \frac{\det\left(\Sigma^{-\frac{1}{2}} W_1 (\Sigma^{-\frac{1}{2}})^T\right)}{\det\left(\Sigma^{-\frac{1}{2}} W_1 (\Sigma^{-\frac{1}{2}})^T + \Sigma^{-\frac{1}{2}} W_2 (\Sigma^{-\frac{1}{2}})^T\right)}$$

$$= \frac{\det Z_1}{\det(Z_1+Z_2)} \quad , \text{ where}$$

$$Z_1 \sim W_d\left(m_1, \Sigma^{-\frac{1}{2}} \Sigma (\Sigma^{-\frac{1}{2}})^T\right) = W_d(m_1, I_d)$$

$$Z_2 \sim W_d\left(m_2, \Sigma^{-\frac{1}{2}} \Sigma (\Sigma^{-\frac{1}{2}})^T\right) = W_d(m_2, I_d)$$

$$\Rightarrow \frac{\det W_1}{\det(W_1+W_2)} \text{ does not depend on } \Sigma$$

* this assumption is not necessary