2.22 $\quad W \sim W_d(m, \Sigma), \quad \Sigma > 0$

$$\frac{\det W}{\det \Sigma} = \det \Sigma^{-\frac{1}{2}} \det W \det \Sigma^{-\frac{1}{2}} = \det\left(\Sigma^{-\frac{1}{2}} W (\Sigma^{-\frac{1}{2}})^T\right)$$

$$= \det Z, \quad Z \sim W_d\left(m, \Sigma^{-\frac{1}{2}} \Sigma (\Sigma^{-\frac{1}{2}})^T\right) = W_d(m, I_d)$$

$$= z_{11} \tilde{z}_{22} \cdots \tilde{z}_{dd}, \quad \text{cf. formula (2.61)}$$

$$\tilde{z}_{kk} = z_{kk} - \underset{\sim}{z}_{k-1}^T Z_{k-1}^{-1} \underset{\sim}{z}_{k-1}, \quad Z_1 = [z_{11}]$$

$$Z_k = \begin{bmatrix} Z_{k-1} & \underset{\sim}{z}_{k-1} \\ \underset{\sim}{z}_{k-1}^T & z_{kk} \end{bmatrix}$$

$$k = 2, \ldots, d$$

Corollary 2 to theorem 2.2 shows that

$z_{11} \sim \chi^2(m)$

Lemma 2.6 and 2.7 shows for $k = 2, 3, \ldots, d$ (cf. lecture note MA5) that

$\tilde{z}_{kk} \sim \chi^2(m-k+1)$ indep. of Z_{k-1} $\quad *$

$\Rightarrow$ $z_{11}, \tilde{z}_{22}, \ldots, \tilde{z}_{dd}$ are independent χ^2 distributed variables with degrees of freedom $m, m-1, \ldots, m-d+1$

$*$ Alternatively by use of lemma 2.10:

$$\tilde{z}_{kk} \sim W_1(m-k+1, 1 - \underset{\sim}{0}^T I_{k-1}^{-1} \underset{\sim}{0}) = \chi^2(m-k+1)$$

indep. of Z_{k-1}

2.23 $W \sim W_d(m, \Sigma)$, $\Sigma > 0$, $m \geq d+2$

i $\dfrac{\underset{\sim}{\ell}^T \Sigma^{-1} \underset{\sim}{\ell}}{\underset{\sim}{\ell}^T W^{-1} \underset{\sim}{\ell}} \sim \chi^2(m-d+1)$ for all $\underset{\sim}{\ell} \neq \underset{\sim}{0}$ cf. lemma 2.7

$$\Rightarrow E\left[\left(\frac{\underset{\sim}{\ell}^T \Sigma^{-1} \underset{\sim}{\ell}}{\underset{\sim}{\ell}^T W^{-1} \underset{\sim}{\ell}}\right)^{-1}\right] = \frac{1}{(m-d+1)-2} = \frac{1}{m-d-1}$$

$$\Rightarrow E\left[\underset{\sim}{\ell}^T W^{-1} \underset{\sim}{\ell}\right] = \frac{1}{m-d-1} \underset{\sim}{\ell}^T \Sigma^{-1} \underset{\sim}{\ell}$$

$$\Rightarrow \underset{\sim}{\ell}^T E[W^{-1}] \underset{\sim}{\ell} = \underset{\sim}{\ell}^T \frac{\Sigma^{-1}}{m-d-1} \underset{\sim}{\ell}$$

$$\Rightarrow E[W^{-1}] = \frac{\Sigma^{-1}}{m-d-1}$$

ii $y \sim N_d(\underset{\sim}{0}, \Sigma)$, y og W independent

$$E[y^T W^{-1} y] = E[\mathrm{tr}(y^T W^{-1} y)] = E[\mathrm{tr}\, W^{-1} y y^T]$$

$$= \mathrm{tr}\left[E W^{-1} E[y y^T]\right] = \mathrm{tr}(E W^{-1} \mathrm{Var}\, y)$$

$$= \mathrm{tr}\left(\frac{\Sigma^{-1}}{m-d-1} \Sigma\right) = \frac{\mathrm{tr}\, I_d}{m-d-1}$$

$$= \frac{d}{m-d-1}$$

2.24 $\underset{\sim}{x}_1, \underset{\sim}{x}_2 \sim N_d(\underset{\sim}{0}, \Sigma)$ independent

$$a\underset{\sim}{x}_1\underset{\sim}{x}_1^T + b\underset{\sim}{x}_1\underset{\sim}{x}_2^T + b\underset{\sim}{x}_2\underset{\sim}{x}_1^T + a\underset{\sim}{x}_2\underset{\sim}{x}_2^T = [\underset{\sim}{x}_1 \ \underset{\sim}{x}_2]\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} \underset{\sim}{x}_1^T \\ \underset{\sim}{x}_2^T \end{bmatrix}$$

$$= X^T A X$$

Corollar 1 to theorem 2.4 shows

$$X^T A X \sim W_d(r, \Sigma) \Leftrightarrow A^2 = A$$

$$\Leftrightarrow \begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} a^2+b^2 & 2ab \\ 2ab & a^2+b^2 \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} a = a^2+b^2 \\ b = 2ab \end{cases}$$

$$\Leftrightarrow (a,b) = \begin{cases} (\frac{1}{2}, \frac{1}{2}) & (r=1) \\ (\frac{1}{2}, -\frac{1}{2}) & (r=1) \\ (1, 0) & (r=2) \end{cases}$$

2.25 $x_i \sim N_d(\underset{\sim}{0}, \Sigma)$, $i = 1, \dots, n$, independent

$X = [\underset{\sim}{x}_1 \ \dots \ \underset{\sim}{x}_n]^T$

A_j, $j = 1, \dots, r$, sym. og idempotent $d \times d$

$A_j A_k = 0$, $j \neq k$

Corollar 2 to theorem 2.4 shows

$$A_j A_k = 0, \ j \neq k \Leftrightarrow \left.\begin{cases} X^T A_j X \sim W_d(m_j, \Sigma) \\ X^T A_k X \sim W_d(m_k, \Sigma) \end{cases}\right\} \text{indep.}$$

$\Rightarrow X^T A_j X$, $j = 1, \dots, r$, mutually independent